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Design of Reinforced Concrete Shells and Folded Plates

P.C. Varghese

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DESIGN OF REINFORCED CONCRETE SHELLS AND FOLDED PLATES
P.C. Varghese

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FOREWORD

It gives me great pleasure to write this Foreword for the book *Design of Reinforced Concrete Shells and Folded Plates* by Professor P.C. Varghese.

I have known Professor Varghese for the past 25 years from the day he joined as a Honorary Visiting Professor in the Division of Structural Engineering, Department of Civil Engineering, Anna University, Chennai in 1984. His professional career spans over six decades. He had the privilege of working as one of the founding staff members of IIT Kharagpur from 1950 to 1961. He headed the Department of Civil Engineering, IIT Madras from 1961 to 1972. He also served the Moratuwa University from 1972 to 1982 and was the UN Advisor to the Ministry of Works, Sri Lanka from 1982 to 1984.

Professor Varghese has already published six textbooks on various subjects. He had studied Shell Structures at the Imperial College, London and also taught the subject at IIT Kharagpur and IIT Madras. Presently, many colleges avoid teaching the subject 'Shells' because of its high mathematical content. He has written this book with much less mathematical content and more of conceptual and practical application. The book covers a wide panorama from the historic development of modern shell roofs to typical designs of folded plates and shells. Several classifications of shells, such as paraboloid shells, conoidal shells, groined shells and folded plates, domes and folded plates are illustrated and discussed in detail.

This book also gives the detailing of cylindrical shells, which is a unique feature. The details of reinforcements according to the conventional system and the necessity of providing the minimum steel at specified spacing are highlighted. The construction aspects including detailing, casting, curing and removal of forms are also explained.

The recommendations of Indian Standard 2210 on the design of pyramid roofs are discussed in the last chapter. There are many worked-out examples which are presented at the end of each chapter. A short history of masonry domes, funicular shells, geometric curves, and tension structures are also presented in the Appendices.

The book is well organized and comprehensive, and provides an exhaustive coverage of the basic concepts and fundamentals to detailed design and construction of the *Concrete*

Shells and Folded Plates. This book will not only prove to be a useful textbook to the postgraduate students but also as reference to a large number of fresh graduates. Further, the book will be a valuable reference to young teachers who teach in various engineering colleges in India. It will also serve as a handbook for practising Civil Engineers.

I congratulate Professor P.C. Varghese for his tremendous effort in writing this book and I commend this well-written and accessible book to all readers—students and practising engineers alike.

Dr. P. MANNAR JAWAHAR
Vice Chancellor
Anna University, Chennai

PREFACE

This book is meant as an introductory textbook on the commonly used reinforced concrete shells and folded plates for the postgraduate students in Civil Engineering and Architecture. The aim is to make them familiar with the planning and design of these modern structural forms. The book explains the action of the various types of these structures, focussing more on the conceptual approach than on a purely mathematical approach so that the students can understand the subject more easily. After going through this book, practising engineers will also be able to undertake the design and construction of moderately sized shells and folded plates with ease. A deeper theoretical knowledge of the subject can be gained by studying the literature given as References provided at the end of each chapter and in the Bibliography at the end of the book.

Another objective of this book is to explain the action and design of different types of concrete shells and folded plates to those who use computer software for their design. This book will give them a better understanding of the outputs and also enable them to make the necessary rough check of the computer output by routine arithmetic calculations, which is always recommended in all practical computer aided designs. The neglect of such conceptual checks has been the cause of many recent structural failures.

The subject of concrete shells and folded plates, known as "Concrete Stressed Skin Structures" became popular only after World War II. Hence, as explained further in the Introduction, they can be called **modern structures**, different from the traditional "concrete columns and slab structures". It is essential that all students specializing in Structures or Architecture should have a fair knowledge of these modern structural forms also. Hence, it is high time the subject was made a compulsory subject in the university curriculum, at least for those specializing in the structural aspects of buildings. I hope this book will help to make this possible. For classroom teaching, the syllabus should not concentrate too much on cylindrical shells, which is covered in the earlier chapters of the book. A general study of all the types of shells and folded plates should be planned so that they can be covered by the

course. Folded plates and other shells, which can be formed by simple form work, are more popular nowadays than curved shells, for example, cylindrical shells.

I will be happy if this book serves as an introductory book to create some interest in the study of modern shell and folded plates among teachers, students and practising engineers.

Any constructive suggestions for improving the contents of the book will be highly appreciated.

P.C. Varghese

ACKNOWLEDGEMENTS

I wish to express my gratitude to the authors, individuals and institutions for their help and assistance in the publication of this book.

As the book has been planned as a textbook for students, I have taken the help from a very large number of textbooks, technical papers and other publications. Even though references to these have been given in the various chapters, I wish to express here my indebtedness to the authors of these publications.

I am specially thankful to the American Society of Civil Engineers for giving me the permission to reproduce the tables from their Manual No. 31 on Design of Cylindrical Concrete Shell Roofs.

Many of the worked examples, without which the book would have been incomplete, are based on the "Notes for a Short Course on Concrete Shells and Folded Plates for Practising Engineers", prepared in 1972 by the Civil Engineering Department, IIT Madras, with Prof. P.S. Rao as the editor. Details of these notes are presented in the Bibliography.

I owe a great deal to Anna University, Chennai for continuing my term as Honorary Professor. All the staff of the Structural and Soil Mechanics Divisions have encouraged me in my work. In particular, Professor G.M. Samuel Knight has been helpful in many ways to get this book finalised for publication. I am very thankful to the Vice Chancellor, Professor P. Mannar Jawahar for agreeing to write the Foreword for this book.

My thanks are due to Mrs. Rajeswari Sivaraman as well her son S. Balaji for typing my manuscript, to Mrs. R. Uma for drawing the figures, and the firm Radix Designs for their formating of these figures.

It has been a pleasure to work with PHI Learning, the publisher of this book. To each of the editorial, production and marketing personnel I worked with, I wish to extend my sincere thanks.

P.C. Varghese

INTRODUCTION

All buildings are meant to enclose spaces. Most of the different forms of superstructures we commonly use for our present-day buildings are only a modification of the age-old system of column, beams and roof covering arrangements. They fulfil their function by using two separate systems. One is the space covering system to cover the space, such as concrete slab or "roof covering sheets" in steel buildings. These are supported by a second system of beams and columns which we may call the supporting system. In many steel buildings, they are obviously separate and in R.C. buildings also, they are treated as two separate systems.

In reinforced concrete shells and folded plates, however, the two functions of covering the space and supporting the covering system are integrated into one. The structure covers the space without beams and columns within the buildings. Thus, they are unique in character and are termed *stressed-skin structures*, different from the traditional ones. Reinforced concrete shells and folded plates are quite modern as the development of these took place only after World War II. Hence, it is essential that modern curricula of Indian universities and engineering institutes in Civil Engineering and Architecture include at least an elementary study of these modern structures.

This book is an attempt to explain the behaviour and design of these "stressed-skin structures" without much derivation of the theory so that all students in Civil Engineering and Architecture can easily follow the subject without much difficulty. As the book is also meant for the young teachers to prepare their lectures, and for the students to revise their lessons, it is lecture-based; it is divided into 20 chapters. There are many other books on the subject available in India giving the mathematical theory. References to them have been made extensively in this book. These can be used for further study of the subject.

Chapters 1–3 give an introduction to the subject. Chapter 4 discusses the membrane analysis of circular domes and cones. Chapters 5–10 explain the bending analysis of circular cylindrical shells with reference to Manual No. 31 of the American Society of Civil Engineers. Design by use of tables given in Manual No. 31 is assumed to make matters simple as it corresponds to the familiar classical methods of analysis of indeterminate structures.

Chapters 11–14 focus on paraboloids, conoids and groined shells. Paraboloids and groined shells have been explained with reference to Parme's paper on that subject. Conoids have been explained with reference to the Manual published by Cement and Concrete Association, U.K.

Chapters 15–18 deal with the design of folded plates. As there are easy methods to build these structures as cast *in-situ* or as precast, they are bound to become very popular in the near future. All over the world, folded plate construction is replacing cylindrical shell construction. Chapters 19 deseribes buckling of shells and Chapter 20 design of pyramidal roofs.

Appendices on the history of domes built before the advent of reinforced concrete, funicular shells, equations of conics, modern tension structures, Tables of ASCE Manual No. 31, and a bibliography on shells and folded plates are also given.

R.C. shells consist of the real shell part and edge members. In the real "shell part" of R.C. shell roofs, the major forces are the membrane forces which are mostly in compression or in low tension. Their design and construction are rather easy. For a successful project, it is very important that a serious thought be given to the analysis of the edge members and the connection between the shell and the edge members. It is important to note that most of the shell construction failures that have occurred in the field have been due to the inadequacy of the design and detailing of the edge members and the supporting members. Their design has been fully explained in this book for the different types of shells.

It is hoped this textbook on the subject, along with the references given for deeper study, will help in the introduction of Reinforced Concrete Shells and Folded Plates in the curricula of Civil Engineering and Architecture in various colleges in India.

P.C. Varghese

Have two goals:
Wisdom (that is, knowing and doing things right) and common sense.
Don't let them slip away,
for they fill you with living energy, and are a feather in your cap.

—Proverbs: 3.21

1

HISTORICAL DEVELOPMENT OF MODERN SHELL ROOFS

1.1 INTRODUCTION

When we plan long span roof structures, our choice usually is between long span steel frames and reinforced concrete thin shells or folded plates. With steel frames, we need additional material like roof sheets to cover the space. Shells or folded plates have the dual property of spanning and also covering the space.

Reinforced concrete thin shells can be defined as curved slabs whose thicknesses are small compared to their other dimensions like radius of curvature. Even though shell construction with steel and concrete is used in the industry for pressure vessels, reactors, etc. we will restrict our study to reinforced concrete shells used for roof construction. Many types of thin shell roof constructions are of recent origin and became popular only after World War II, even though progress was made in reinforced concrete after World War I. This modern material (reinforced concrete) can be cast in any shape. Like steel, it has strength. Besides, it has the body to cover space. Steel by itself is rarely used to cover space whereas reinforced concrete is used to construct slabs, shells, etc. which in addition to spanning lengths can also cover space. This important property of reinforced concrete can be taken full advantage in shell construction.

The concept of shell construction can be considered as a slow evolution made from masonry arches and domes, which were in use from the very early days of human existence. Brickwork could take only compression and no tension. Reinforced concrete is homogeneous and continuous and its action is different from that of brickwork made of individual bricks with mortar joints. Thus, the action of a brick arch with mortar joint is different from that of a reinforced concrete arch. Brick arch cannot take bending whereas an R.C. arch can resist bending also. It is a very interesting study as how, without the availability of modern materials, our ancient architects were able to design and construct

structures like large domes and vaults to cover large space for monumental buildings. In this chapter, we focus on the evolution of modern reinforced concrete thin shell structures.

1.2 ADVENT OF BRICK VAULTS

Brick vaults (three-dimensional arches) are the oldest form of curved roof structure. As explained in Appendix A, they can be considered as an extension of the masonry arch. Arches are believed to have been invented by the Greeks around 5th century BC [1]. The use of bricks for roof became a necessity in old days as most of the earlier buildings built with wooden beams and columns tended to be burnt down by fire. Hence, humans invented the vaults made of bricks, which were safe from fire. Though the invention of vaults is generally credited to the Greeks, there is evidence to show that the Mesopotamians and Egyptians had also experimented with it, though sparingly, before the Greeks. But it was the Romans who used arches, vaults and domes extensively.

1.3 EVOLUTION OF BRICK DOMES

Domes which can be looked upon as rounded vaults with circular or polygonal bases were the next to be evolved. Vaults are easy to build, but domes require much more skill and improved materials. Domes are three-dimensional structures subjected to special forces. The brick domes were invented by the Romans. They were fond of the circle and used circular domes very freely. The dome is three-dimensional in nature and has two types of stresses (meridinal and hoop stresses) instead of one as in a vault. It is as different from a vault as a plate is from a beam when used to span an area. This improved structural action of domes was used by the Romans for large structures like the Pantheon. However, these brick domes were massive structures. The Pantheon brick dome, about 43 m (142 ft 6 in) in diameter built around 150 AD by the Romans, had its central portion about 1.2 m (4 ft) thick. Most of these domes had a central thickness of 0.025 times its diameter whereas a modern 50 m diameter R.C. dome needs only 60 to 70 mm in thickness. (A minimum thickness of 100 mm is usually adopted for placement of steel in all R.C. shells.) From modern analysis, we know that in a concrete dome only a fully semicircular dome has compatibility with support conditions and that they develop hoop tension at points below about 51°.48′ (say, 52°) below the central half angle. They also develop horizontal thrust at the base when built as segments less than 90° central half angle as shown in Figure 1.1. These principles were not fully known in early days. The lack of this knowledge of forces acting on a dome caused a large number of failures in domes. However, it is interesting to study how old master builders solved the problems. A short statement of this aspect is given in Appendix A.

It is worth mentioning that the building material which brought forth the structural innovation of the Romans was 'Roman concrete' described in Section 1.11.

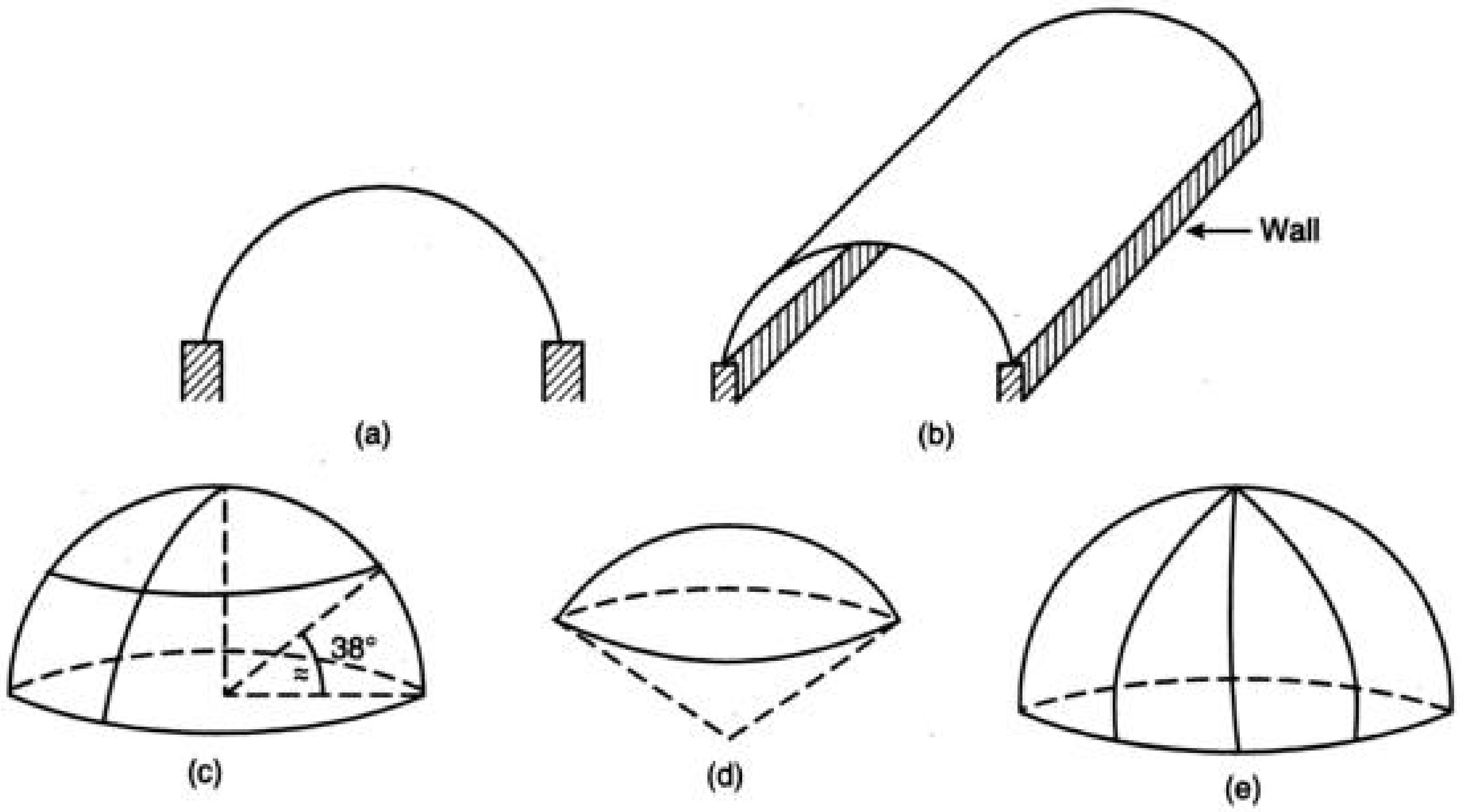

Figure 1.1 Evolution of domes: (a) Arch, (b) Vault supported on wall, (c) Circular hemispherical dome indicating tension below about 38° as shown, (d) Circular segmental dome, and (e) Ribbed dome.

1.4 RIBBED DOME CONSTRUCTION

Because of liable tension in semicircular domes and horizontal thrust in segmental domes, the ancient and Middle Ages builders resorted to special methods of construction. One of the methods was to build a number of arches spanning the diameter and the space, in between filled with suitable materials like brickwork. For example, the dome in St. Andrews Church in Chennai on Poonamallee High Road in Egmore built by the British in 1821 (before reinforced concrete became popular) was constructed with brick arches spanning across the diameter, with the space in between the arches filled with the Sicilian cones. (These are clay cones like the modern ice cream cones fitted into each other and arranged in rows between the arches and plastered, thus reducing the weight of the filling between arches.) This structure is still being used for worship in Chennai (See Appendix A for description of this dome).

There are many other structures in the world which were built as ribbed domes (described in Appendix A). Even as late as 1911, the dome of the Melbourne Public Library in Australia was built of ribbed dome construction. Many famous domes of the world in masonry were of this construction.

1.5 ADVENT OF REINFORCED CONCRETE SHELLS

Though Lombart of France built a boat out of steel mesh and concrete (reinforced concrete) as early as in 1850, it was only just before World War I that reinforced concrete came into use. (It is interesting to note that prestressed concrete became popular after World War II.) The aircraft hangar at Corley in France built by Fressinet around 1917 shown in Figure 1.2 can be considered as the first real reinforced concrete shell construction. It was in the

form of arches spaced at small spacings. The catenary shells spanned between the arches. The arches were constructed from the ground (foundation) level so that the reactions are taken by the ground. These arches were connected together in the longitudinal direction by corrugations (shells) as shown in Figure 1.2 [1].

Figure 1.2 The aircraft hangar at Orly, France, built in 1917 (designed by E. Freyssinet).

The mathematical theory of shells was not known to engineers till the membrane theory of shells (usually credited to G. Lame and G. Clapeyron) was published in 1828 (19th century). However, it was only as late as in 1923, about 100 years later, that Dischinger and Baversfeld used the membrane theory to build a reinforced concrete dome over the planetarium in Jena. It was also Dischinger who, in 1928, brought to the notice of design engineers the concept of membrane theory for the design of shells. Another dome that Dischinger and Baversfield built is a hemispherical dome, only 30 mm thick and 16 m in diameter. There is no horizontal reaction as the ends are in 90° but it develops hoop tension below 52° which is resisted by steel reinforcement.

Engineers soon realized that reinforced concrete, unlike steel, can be made continuous and used to cover space. Reinforced concrete has both strength and body and is a much superior material to cover large space in building construction. Steel and wood can be used as beams and columns but not as slabs or shells. With prestressing, we can cover very large spaces. This concept has encouraged the present use of shells for covering very large areas like factories, hangars, assembly halls, etc. With the rising cost of steel and concrete, shell construction is bound to become popular for large span roofs.

1.6 THEORETICAL ADVANCES

Much progress was made in mechanics during the period around 1800 AD. The use of iron and steel brought in the method of theoretical design of columns and beams by engineers trained in mechanics to replace artisans trained in traditional design by thumb rules obtained from experience in wood and brickwork construction. As already stated, the membrane theory was applied to design of concrete shells only as late as 1923 by F. Dischinger. He also wrote a special chapter on cylindrical concrete shells in the German Handbook Eisenbetonban with U. Finsterwalder in 1928 [1]. The membrane

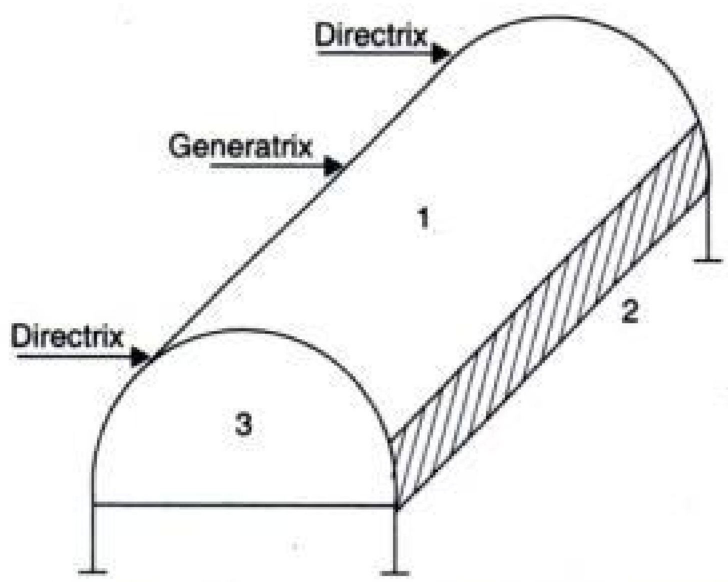

Figure 1.3 Cylindrical shell: (1) Shell, (2) Edge beam, and (3) Transverse (Transverse can be a solid beam or a tied arch or a rigid frame).

theory is a statically determinate solution and it advanced very much the analysis and design of R.C. shells.

At first, interest in Western countries was limited to the design of cylindrical shells as shown in Figure 1.3. This was initiated by F. Dischinger and U. Fisterwalder. As bending is also predominant in cylindrical shells, the bending theory of shells was evolved. However, solution to the bending theory is possible only with a number of simplifications. These were made by H. Schorer in 1935 [2] and also by R.S. Jenkins in 1947 [3]. The American Society of Civil Engineers published a manual on 'Design of Concrete Shell Roofs' in 1952 [3]. The theory for design of many other types of shells was proposed in later years. Here, we, however, discuss only the basics of the subject to understand the fundamentals. Today, many computer softwares are available for the design of shells which has made this construction very popular. The aim of this book is not to describe the mathematical theories but to give a conceptual idea of the forces that act on the various types of shells so that we can understand and interpret the computer output meaningfully.

1.7 COMPARISON OF OLD BRICK DOMES WITH MODERN R.C. DOMES

The use of reinforced concrete has considerably reduced the weight of materials per square metre of area used in the construction of domes as can be seen from Table 1.1.

TABLE 1.1 Comparison of Weights of Material Used [1]

S.No.	*Building*	*Construction*	*Date of Construction*	*Span, in feet*	*Weight/sq. yard* (lbs)	*Percent*
1	St Peter's Dome	Masonry dome	1590	137	12,500	100
2	Centenary Hall, Breslaw	Ribbed R.C. dome	1912	215	3750	30
3	Market hall, Algeciras	Reinforced concrete shell dome	1933	160	480	3.8
4	Schawrzwaldhalle (Karlsruche)	Prestressed concrete saddle shell	1953	240	250	2.0
5	Le Toureau hemisphere, Texas	Aluminium steel dome	1953	300	41	0.33
6	C.N.I.T. Exhibition Hall, Paris	Double reinforced concrete shell dome	1958	720	520	4.1

1.8 DEVELOPMENT OF RULED SURFACES AS SHELL ROOFS

The high cost of curved formwork is a major objection to the use of curved shells like circular domes and cylindrical shells in general. This has led to the development of ruled

surfaces for shells, like hyperbolic paraboloids and conoids, shown in Figure 1.4, for roof construction. For these thin structures, linear forms of formwork can be used for construction. The reduction of the quantities of steel and concrete needed for shells as compared with ordinary slabs and columns is sure to make shells popular for covering of large spaces in the future. Hence, their study is important for any structural engineer.

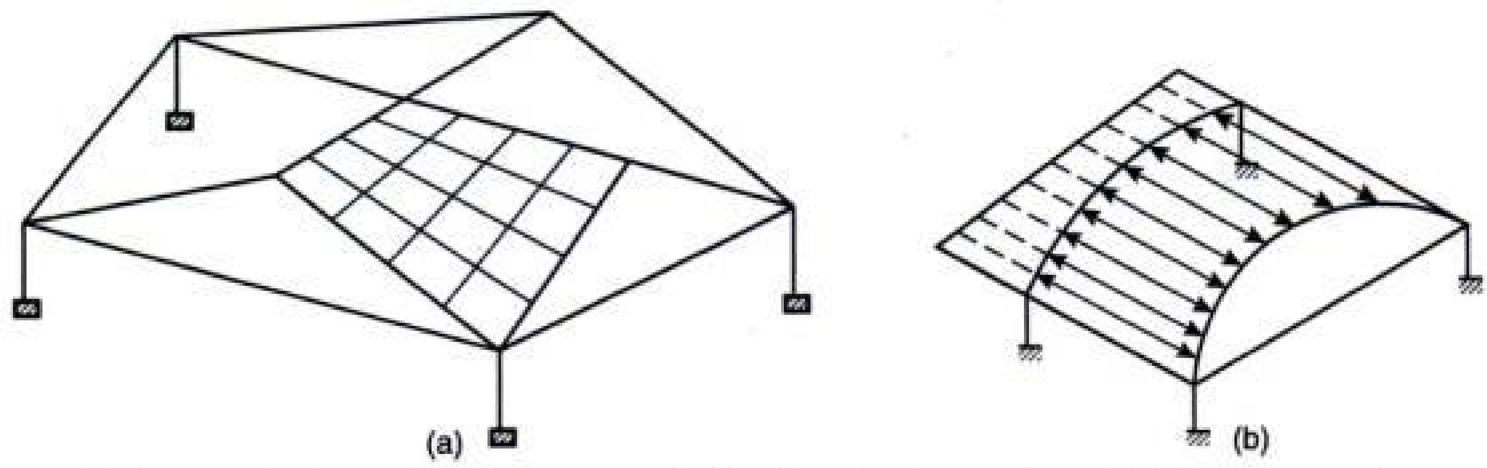

Figure 1.4 Ruled surfaces: (a) Hyperbolic paraboloid (doubly ruled surface), and (b) Conoid (singly ruled surface).

1.9 DEVELOPMENT OF FOLDED PLATES FOR ROOFING

During the early 1920s, coal bunkers and similar containers were built in Germany without beams at the junctions of the plates (Figure 1.5). The concrete slab is folded and the reinforcement steel is especially detailed for continuity. This led to the development of the conception of folded plates, as shown in Figure 1.5, for roof construction also. The folded plate behaves as a slab in the horizontal direction and as a plate or beam in the vertical plane. This combined slab-plate action makes it a very economical roof structural element. The use of moment distribution method to analyze these plates was proposed by G. Winter and M. Pei in 1947 [4]. Further refinement of the analysis by Yitzhaki [5] enables us to arrive at a good analysis of these structures [5]. The ACI publication by a task committee has given us a very good method to analyze these structures [6]. This has made their analysis and design easy. Such construction became very popular in use as the formwork of this type of construction is simple (Figure 1.4). Today, in many factories, instead of cylindrical shells, folded plates are used. Even the use of prefabricated folded plates cast on ground and lifted up in position is also popular in the construction of roofs over large areas.

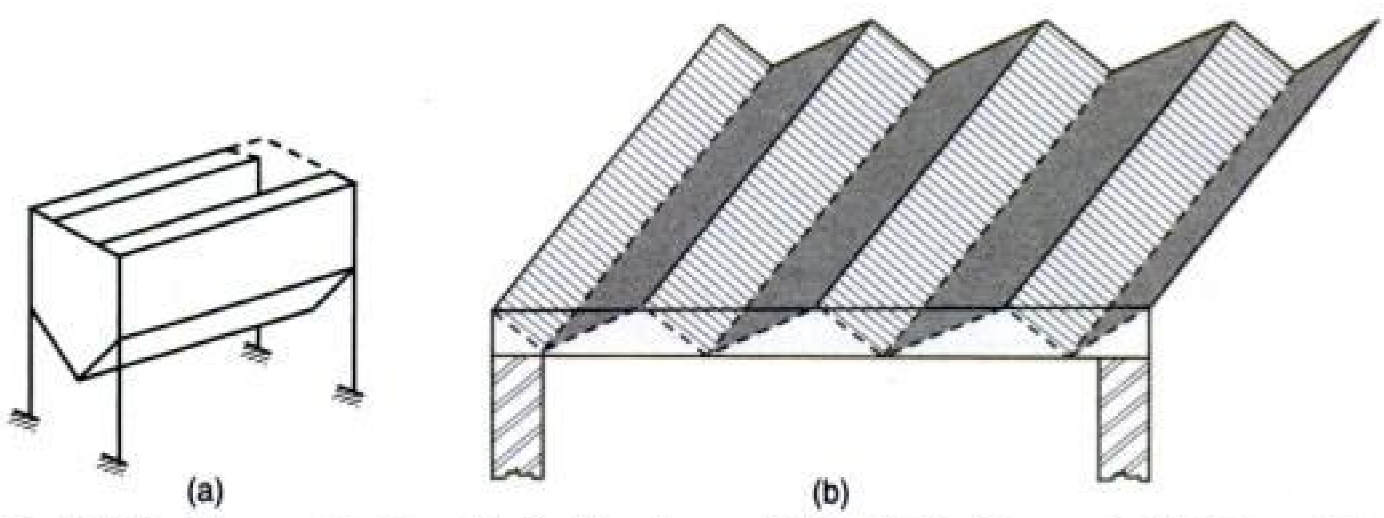

Figure 1.5 Folded plate construction: (a) Coal bunker, and (b) Folded plate roof with 'v' type folded plates.

1.10 GROINED VAULTS

The term 'groin' in architecture means "edge formed by intersecting arches" as shown in Figure 1.6. Intersecting surfaces made of parabolic arches are very beautiful to look at and are very popular in use. These groined vaults are used to cover large spaces. The chief difference between conventional shells and groined vaults is the edge effect. In groined vaults, as one curve merges into the other, there are very little edge disturbances in these thin shells (In fact, some of the sea shells like the one used for advertisement of shell oil company occur naturally in this shape). These shells are widely used at present for covering large spans and are also amenable to conceptual design using the theory of arches.

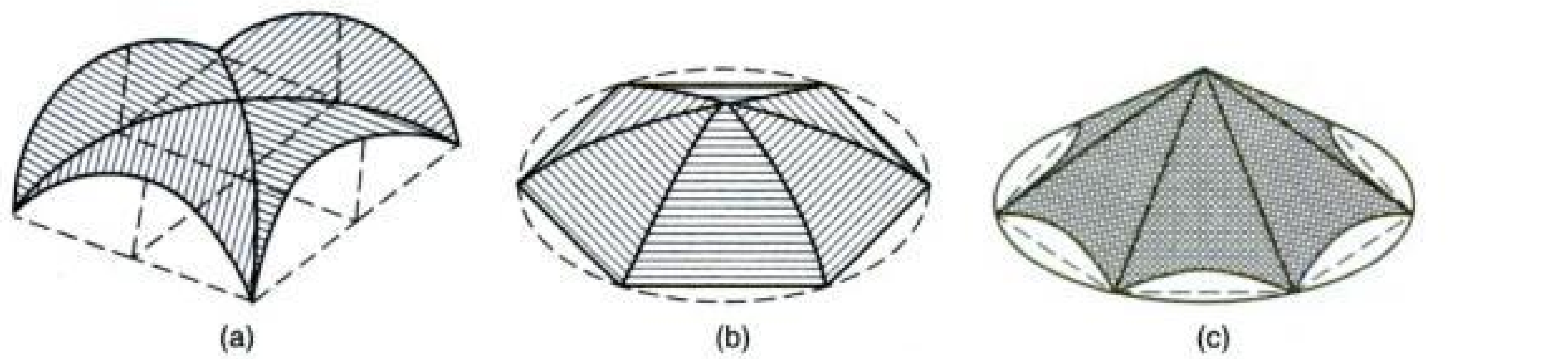

Figure 1.6 Composite shells: (a) Groined shell with various types of translational (cylindrical or hyperboloid) shells, (b) Polygonal dome composed of individual cylindrical shells, and (c) Combination of skew hyperbolic paraboloids over a polygonal plan.

1.11 PROGRESS IN MATERIALS OF CONSTRUCTION

We must remember that the building material which led to the great structures of the Romans was the invention of Roman concrete [7]. Very few engineers are aware that it is this strong material that enabled the Romans to build vaults, domes, etc. of great beauty and ease. The important ingredient was pozzolane, a volcanic earth found in the thick strata in and around Rome. Romans mixed this pozzolanic earth with lime which, when wetted, produced a very strong mortar. (This mortar sets under water like modern cement.) What is now known as Roman concrete consisted of layers of these mortars into which rubble was firmly pressed in. It was different from what we do today when cement mortar and rubble are mixed together in a mixer before being placed in position. It was this Roman concrete that the Romans used in many of their important constructions. In modern times, the advancement made with reinforced concrete by introduction of prestressing after World War II fibre concrete etc. has made it possible to make great progress in shell construction.

1.12 MODERN COMPUTER METHODS OF ANALYSIS OF SHELL STRUCTURES

In modern times, a large number of software are available for easy analysis of all types of structures, including shell structures. However, unless one has a full physical

understanding of the behaviour of each type of shell, it is difficult to check the output of the computer and imagine its magnitude. Also, some rough methods should be also available to check these computer results.

Earlier, the structural analysis of indeterminate structures depended on the analysis of deformation. Later, matrix methods based on stiffness concepts evolved. This has now been replaced by computer software mainly based on finite element analysis. In our discussions in this book, we depend more on conceptual behaviour and deformation analysis than on theoretical derivations to enable us to visualize the magnitude and nature of the computer output.

SUMMARY

This chapter gives a brief description of the evolution of the modern shell structures. Nowadays, shells of various shapes are very much used for covering large areas [8]. As concrete has both body and strength, it is the first choice for constructing very large exhibition halls, churches and other places of assembly. Another choice for large span structures is three-dimensional steel trusses. However, any steel structure behaves only as supporting structure. Additional materials like aluminium sheets are required to cover the space. In this book, we discuss the design of important types of reinforced concrete shells and folded plates generally used in practice.

REVIEW QUESTIONS

1. State the period (years) in human history when the Roman civilization lasted. Define Roman concrete and state how it helped the Romans to build improved structures.
2. When did reinforced concrete construction come into common use? State how the introduction of reinforced concrete helped engineers in building construction.
3. When was prestressed concrete introduced into Civil Engineering practice? How has it helped the planning of long span structures?
4. Sketch the following structural forms: (i) Arch, (ii) Vault, (iii) Dome, (iv) Hyperbolic paraboloid, and (v) Folded plate construction.

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2

COMMON TYPES OF SHELL ROOFS AND DESIGN BY WORKING STRESS METHOD

2.1 INTRODUCTION

We have seen that the shell is a thin curved slab whose thickness is small compared to its other dimensions like span, and radius of curvature. Usually, the thickness/radius ratio (t/r) will be 1/100 to 1/500. In nature, we can find shells of all shapes and dimensions, such as sea shell and, egg shells. Generally, they are curved in both axes and do not form a definite geometric pattern. It is said that "Perhaps one of the most beautiful structural shape in nature is the shell of a bird's egg". However, in the mathematical theory of shell structures as applied to engineering design, we have to restrict our study only to shells that are formed by definite geometrical shapes such as circle, parabola, hyperbola, and ellipse. In this chapter, we briefly describe common types of shells used for roofs in civil engineering practice. IS 2210–1988 gives a summary of these shells [1].

Shells come under the category of stressed skin structures. The common shapes of shell roofs that we deal in this book are shown in Figure 2.1 to Figure 2.6 and Table 2.1.

We will also review design of reinforced concrete members by working stress method.

2.2 GENERAL CLASSIFICATION OF SHELLS

We generally classify shells into the following classes (Table 2.1):

1. Shells of revolution
2. Translational shells
3. Ruled surfaces
4. Composite shells.

[To these, we will add folded plates. Even though these are not true shells, these are also thin slabs and are usually associated with shell roofs.]

Each class of shells is discussed in the following sections.

TABLE 2.1 Classification of Stressed Skin Roof Structures

- Polygonal domes
- Shell roofs
  - Shells of revolution
    1. Circular domes
    2. Conical domes
  - Shells of translation
    1. Cylindrical domes
       (a) Circular cylindrical domes
       (b) Parabolic cylindrical domes
       (c) Elliptic cylindrical domes
       (d) Catenary cylindrical domes
    2. Hyperboloids
       (a) Hyperbolic paraboloids (bounded by parabola)
       (b) Hyperbolic paraboloids (bounded by straight lines)
       (c) Circular paraboloids
       (d) Elliptic paraboloids
  - Ruled surfaces
    1. Hyperbolic paraboloids (bounded by straight lines)
    2. Conoids
  - Composite shells (Groined vaults)
  - Funicular shells
- Folded plates
- Pyramidal roofs

2.2.1 Shells of Revolution

The middle surfaces of *shells of revolution* are formed by rotation of a plane curve K or a line about the vertical z axis in its plane as shown in Figure 2.1. The generating curve is called the *meridian*. If the meridian is a straight line rotating around a vertical axis, we get a cone. If it is a vertical circle rotating about a vertical axis, we get a sphere. If the meridian is a parabola, we get a paraboloid of revolution, etc. (We will study the analysis of circular domes and conical domes which come in this category.)

2.2.2 Translational Shells

The middle surface obtained by the translation of a moving plane curve K_1, called the generator (*generatrix*), which moves parallel to itself over another stationary curve K_2, called the *directrix*, (the function of these being interchangeable) gives us a translational

shell as shown in Figure 2.2. The directions and the magnitudes of the curvatures $1/R_2$ and $1/R_2$ (radii of curvatures R_1 and R_2) of K_1 and K_2 are very important. The properties of these shells are explained below. Thus, the curves may point in the same or opposite directions. When they are in the same direction, they are called *clastic or synclastic shells*. When they are in opposite directions, they are called *anticlastic shells*. The following shells come in this category.

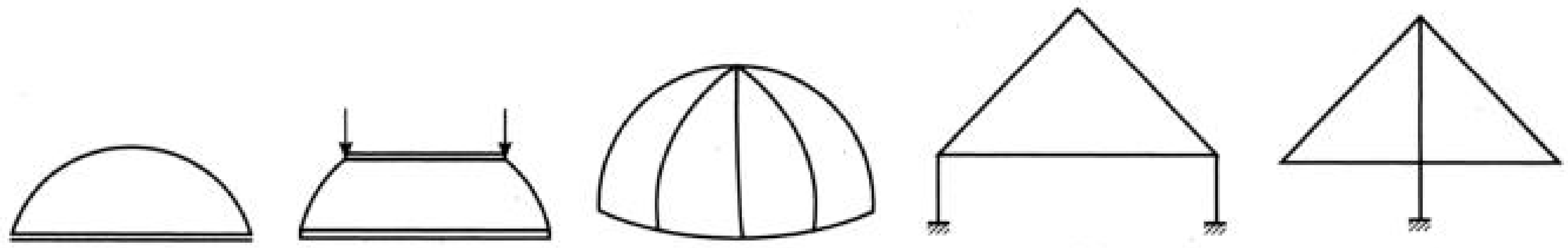
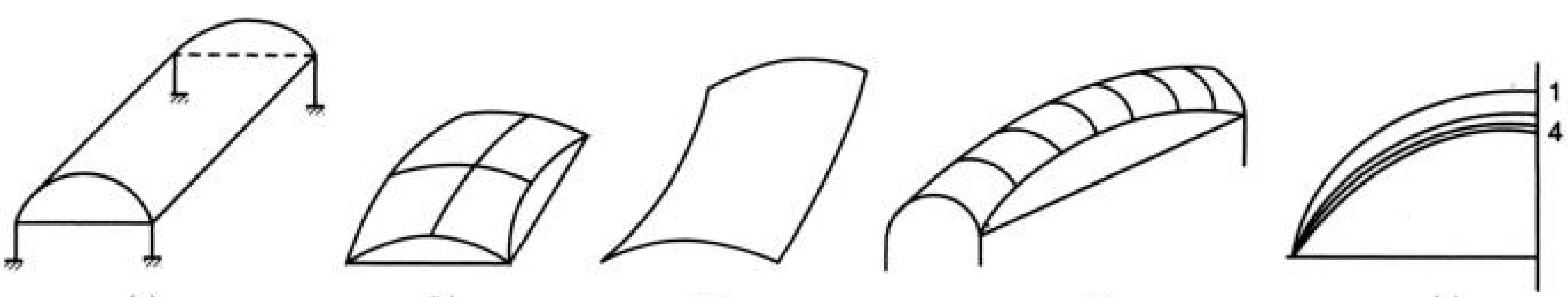

(a) (b) (c) (d) (e)

Figure 2.1 Rotational shells: (a) Simple circular dome, (b) Dome with sky light, (c) Ribbed dome, (d) Conical dome, and (e) Conical umbrella roof.

1. **Cylindrical shells:** Thus [Figure 2.2(a)] are formed by a straight line generator moving along a specified directrix. The common types of cylindrical shells are the following with straight line generators:
   - Circular cylindrical shells—Circular arc directrix
   - Catenary cylindrical shells—Catenary directrix
   - Parabolic cylindrical shells—Parabola directrix
   - Elliptic cylindrical shells—Semi-ellipse

   The layout of the curves will be as shown in Figure 2.1(d).
2. **Paraboloid shells:** As shown in Figure 2.2(b) and (c), a parabola with downward curvature moving on another parabola with curvature in the *same direction* will produce a *circular paraboloid* if the parabolas are equal. It will produce an *elliptic paraboloid* if they are unequal. On the other hand, with two parabolas of *opposite curvatures*, one moving on the other will produce a *hyperbolic paraboloid* (Hypar) shell.
3. **Toroidal shells:** These special shells of revolution have the generatrix of a much sharper curve than the directrix as shown in Figure 2.2(d). These can be used as a series connected as multiple shells to produce very large curved areas like aircraft hangars. Spans up to 100 m have been built with these shells.



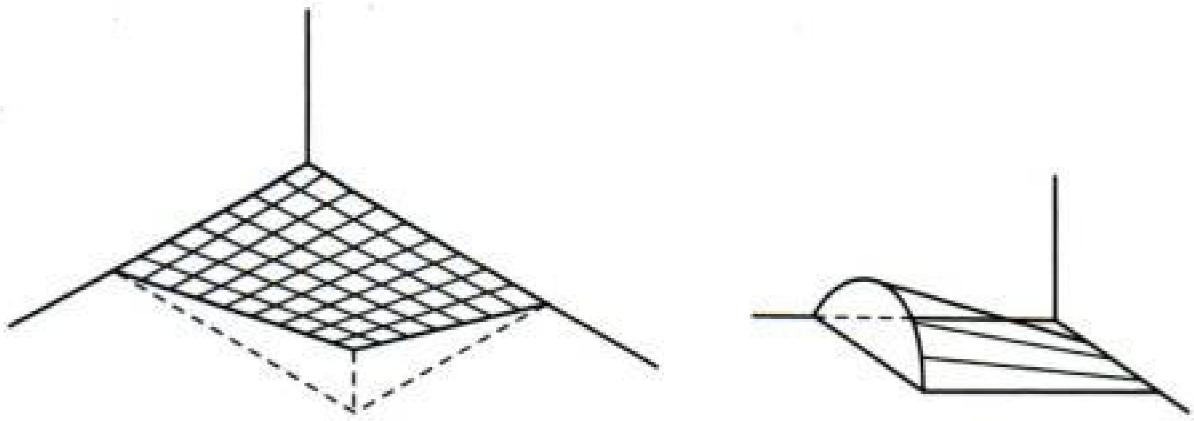
(a) (b) (c) (d) (e)

Figure 2.2 Translational shells: (a) Circular cylindrical shell, (b) Elliptic (or circular) paraboloid shells, (c) Hyperbolic paraboloid shell, (d) Toroidal shell, and (e) Different types of curves—ellipse, circle, catenary, parabola.

2.2.3 Ruled Surfaces (Hypars and Conoids)

Ruled surfaces are surfaces which can be generated entirely by straight lines. They can be singly or doubly ruled surfaces. A surface is said to be singly ruled if at every point on its surface a single, straight line can be drawn along the surface. The surface is called a doubly ruled surface if at every point on the surface two straight lines can be ruled.

It will be shown in the chapters dealing with hypar and conoid shells that though these surfaces are of negative Gaussian curvature, a hypar can also be obtained by a straight line moving over two inclined straight lines and that the conoid can be obtained by a straight line moving over two curves at its ends as shown in Figure 2.3. Conoids are singly ruled surfaces whereas hyperbolic paraboloid is a doubly ruled surface ruled in two directions. Cylindrical shells can also be classified as a singly ruled surface.

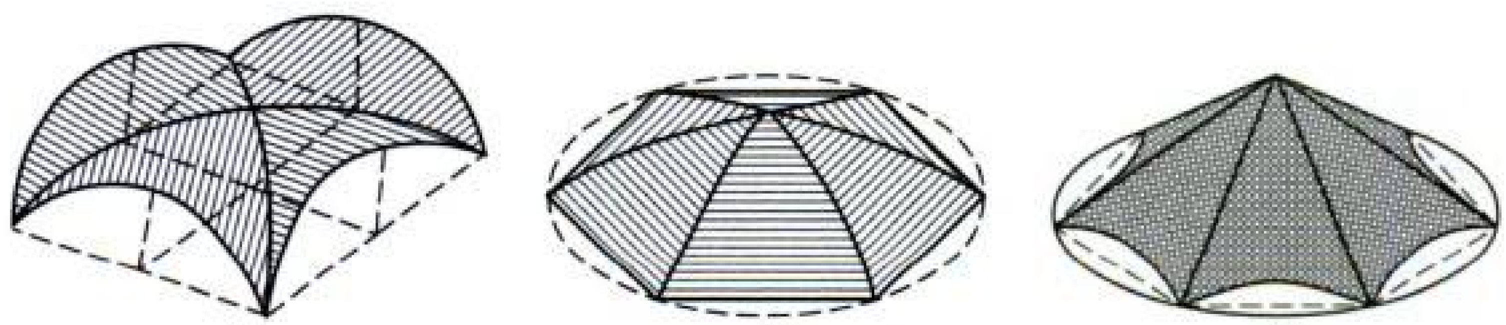


(a) (b)

Figure 2.3 Ruled surfaces: (a) Hyperbolic paraboloid with straight edges (doubly ruled surface), and (b) Canoids (singly ruled surface).

2.2.4 Composite Shells (Groined Vaults)

Composite shells are combinations of two or more shells as shown in Figure 2.4. Individual hyperbolic paraboloid surfaces or cylindrical shells can be joined together to form many surfaces.



(a) (b) (c)

Figure 2.4 Composite shells: (a) Groined shells formed by cylindrical shells or hyperbolic paraboloids, (b) Polygonal dome with cylindrical shells, and (c) Combination of skewed hyperbolic paraboloids.

2.2.5 Folded Plates (Hipped Plates)

We have already seen in Section 1.9 that folded plates (Figure 2.5) are a series of inclined slabs connected together and supported at their ends. Though folded plates are not curved and cannot strictly be called shells, they are thin structures and are usually studied under

the subject of shell structures. These are sometimes called "stressed skin" structures. Nowadays, more of these structures are used in common practice because of their simplicity in formwork.

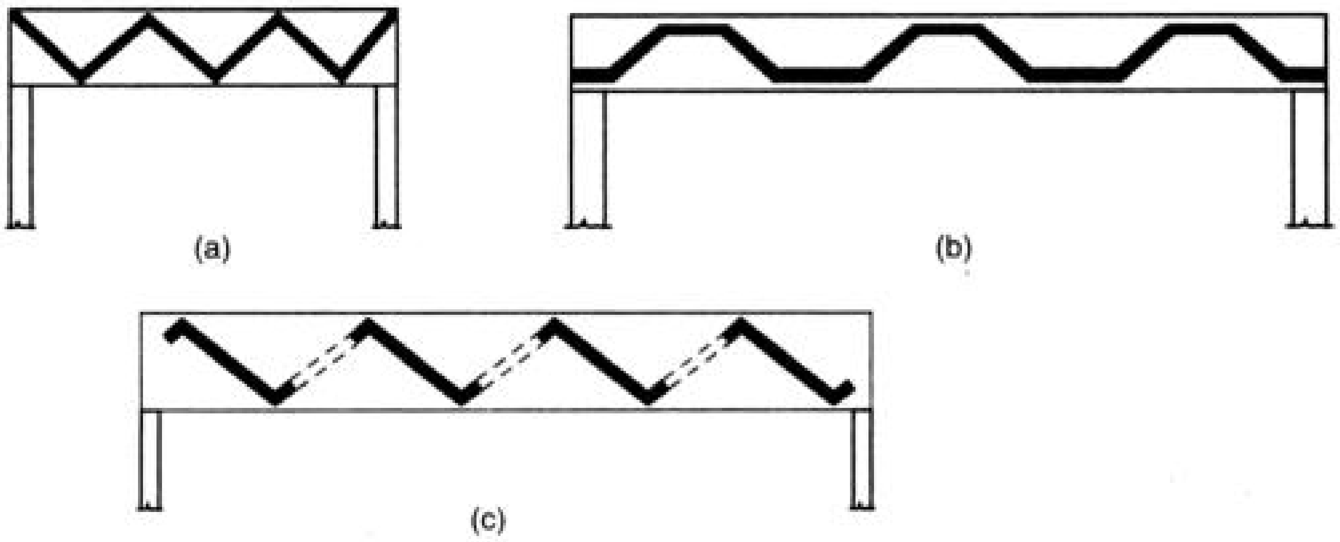


Figure 2.5 Folded plates: (a) V type, (b) Trough type, and (c) Z (northlight) type.

2.2.6 Pyramidal Roofs

Pyramidal roofs, as shown in Figure 2.6, are not pure shell structures. As they are also made of thin slab construction, we will briefly study its design.

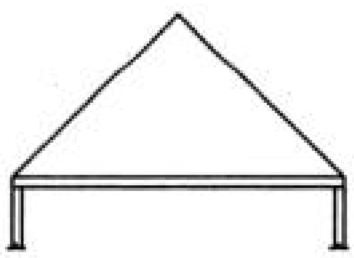


Figure 2.6 Pyramidal roofs.

2.2.7 Catenary and Funicular Shell Roofs

The word 'funicular' means "of a rope or its tension". In most shells, the principal load is the dead weight. If we shape the shell like a *catenary*, the shape a rope will take under its own weight, there will be no bending. (However, this is true for only the dead weight loading.) Resistance of these catenary shells against lateral loads can be improved by corrugating them. Shells of single curvature of the shape of a reversed hanging rope are called *catenary shells*.

Funicular shells are of double curvature. These can be obtained by stretching a piece of hessian on flexible fabric along its boundaries (scaled down to the plan of the building) and loading it with a wet layer of plaster of paris. A short description of these is given in Appendix B. (Details of design is described in References 2 and 4. Reference 2 gives a chapter on funicular shells.)

2.2.8 Gaussian Curvatures

The product of the two principal curvatures with their signs is called the *Gaussian curvature* ($G = 1/R_1 \times 1/R_2$) at any point on the surface of the shell. This can be positive, zero or

negative. The surface is termed synclastic developable or antielastic depending on whether the Gauss curvature is positive zero or negative, respectively. We will see later on that one of the important properties of the shell is given by the Gaussian curvature and its effect on the propagation of edge disturbances in the shell. For shells with positive Gauss curvature like domes, the edge effects tend to dampen rapidly and are confined to a narrow zone of the shell. This makes the membrane theory of these shells (such as domes) valid throughout the shell, except for a small portion near the edge member of the shell. For a shell of zero curvature like the cylindrical or Conical shells, the dampening is not rapid but extends further into the shell than in the case of shells of positive Gauss curvature. However, in shells of negative Gauss curvature like the hyperbolic paraboloids, the boundary effects due to edge members penetrate further into the shell. This information gives us an idea of the extent of reinforcing of the shell portions near edge members. (Circular cylindrical shells are produced by a straight line moving over two equal circles at its ends. Its Gauss curvature is zero. This is the case with conical shells also.)

2.3 TYPES OF SHELLS DISCUSSED

In the ensuing chapters, we will confine ourselves to the following types of shells. Though there are many other types, a study of these shells will give us an understanding of the action of shell structures so that further studies can be made by referring to advanced textbooks on the subject.

1. Circular domes
2. Conical shells
3. Cylindrical shells
4. Hyperbolic paraboloids
5. Elliptic paraboloids
6. Conoids
7. Groined vaults
8. Polygonal domes
9. Folded plates
10. Pyramidal roofs

The aim is to make the readers familiar with the behaviour of these shells. An elaborate analysis will require more exhaustive treatment of these subjects.

2.4 SELECTION OF DIMENSIONS OF SHELLS

IS 2218–1988, Section 7 gives us some guidance in choosing the preliminary dimensions of the various types of shells [1]. We will review it in the various chapters which dealing with various types of shells.

2.5 GENERAL METHOD OF STRUCTURAL ANALYSIS OF SHELLS

In the analysis of shells, we will study their behaviour under two heads, namely, membrane analysis and bending analysis. Such a procedure can be used to understand their basic behaviour. Today, with the availability of computer software, we can solve the problems

much more easily. But the basic understanding is needed to interpret the results for detailing the steel reinforcement and also for designing boundary elements of the shells.

1. **Use of membrane theory:** As a first step, we assume that all the loads are taken by membrane forces or direct forces, namely, direct tension or compression along the radical directions and shear without bending and torsion. As there are only three forces, the three equations of equilibrium will give us all the forces. It is a determinate or primary system.
2. **Determination of errors towards the boundary:** The forces and also displacement got by membrane analysis need not be in equilibrium or compatible with the actual situation at the edges of the structure, where the shell is restrained by edge members. Hence, we find the *errors at* the boundaries due to membrane stress resultants.
3. **Corrections:** Next, we find the *corrections* that have to take place due to unit edge effects (forces and displacement) applied at the boundaries. For this, we used the *bending theory* of the shell.
4. **Compatibility:** Next, we achieve *compatibility* by computing the magnitude of the edge effects necessary to eliminate the errors.
5. **Resultant forces:** The sum of the membrane forces and the compatibility conditions gives the resultant forces.

The above method for a complete solution of shell structures is similar to the analysis of indeterminate linear structure, where we first make the system determinate and then find out the necessary force resultants to make the forces and displacements compatible with the real structure. We will use this method to study the bending analysis of cylindrical shells.

Alternately, we can also derive the general expression for total analysis of shells by matrix methods. General expressions for the combined membrane analysis and bending analysis of all types of shells will be quite involved [2], [3]. Alternately, we can use the modern finite element analysis using computer programs [3]. Instead we will take the analysis of different geometric shapes of shells and show how the membrane analysis of these shells can be made. The bending analysis will be discussed with the help of published Tables for cylindrical shells and hyperbolic conoids.

2.6 STRUCTURAL ANALYSIS

It is easy to make membrane analysis of shells as we are analyzing a determinate structure. Although it is difficult for average structural engineering students to master the mathematical intricacies of the bending theory of shells, it is not so for them to understand the underlying static principles of shell action. By understanding and applying the principles of R.C. design, it is possible for an average student to plan and undertake practical design of *simple shell structures* with as much ease as with many conventional types of structures. We must remember that in many branches of structures, as in the case of flat slab and shells, their construction took place well before the exact theories of shell design were developed. Hence, intuitive design should also be used in all design, detailing and construction of shells.

2.7 STRUCTURAL DESIGN BY WORKING STRESS METHOD

As shells are thin structures, it is considered better to use the *principles of elastic design*. We also use M20 or M25 concrete to have good strength but not too much shrinkage which can produce cracking. The use of Fe 215 steel is ideal for shells. However, we use the readily available Fe 415 (but not Fe 500) for economy. The usual IS456 (Appendix B) recommended design values as given below are to be used.

1. **Loads on shells (see also Section 5.5):** Shells are to be designed for all loads (dead, imposed and wind loads) as specified in IS 875, parts 1 to 3. Generally, we adopt the live load as that specified for "roof without access" equal to 75 kg/m^2 for sloping roof less than 10°. (For roofs sloping more than 10°, it is 75 minus (0–10) kg/m^2).

   Hence, for a shell surface we may use a maximum live load of 75 kg/m^2 plus a water proof 15 kg/m^2.

   Accordingly, a total addition load 70 to 100 kg/m^2 in addition to dead load will be satisfactory (Fischer in his book uses a live load of 40 kg/m^2 and insulation of 12 kg/m^2 a total of 52 kg/m^2. Ramaswamy uses a total live load of only 15 lbs/sqft (≈ 70 kg/m^2 including waterproofing).

   [**Note:** We should note that the live load in always assumed to act on the projected area of the shell surface which is less than the surface area of the shell. Hence when the specified live load is combined with dead load, its value in theory need to be only less than the specified value.]

   Regarding wind loads as shells are curved addition wind load is not generally taken for shells, suction loads will only reduce the stresses.

   [**Note:** Examples in this books use various amounts of live loads.]

2. Allowable direct and bending compression in concrete for design by working stress method (Elastic design)
   IS 456 (Annexure B) gives the following values for elastic design of reinforced concrete.

TABLE 2.2 Allowable Stresses in Concrete (N/mm^2)

Concrete grade	*Direct compression,* (σ_c)	*Bending compression,* (σ_{cb})	*Shear* t_c *in concrete*
M 20	5	7	0.18 to 0.51
M 25	6	8.5	0.19 to 0.57
M 30	8	10	0.20 to 0.60

(From Tables 21 and 23 of IS 456)

[**Notes:**
1. The above direct compression is about 0.25 fck.
2. Allowable compression should be less than the critical stress for buckling (Chapter 19).
3. Value of modular ratio $m = 280/3\sigma_{cb}$ (or 280/fck).
4. Allowable shear depends on percentage of longitudinal steel (IS 456 Table 23). Values given above are for percentages < 0.15 and 2.0.]

3. **Allowable tension in steel (f_s)**
   (a) For Fe 250 steel up to 20 mm dia. 140 N/mm^2 and above 20 mm. 130 N/mm^2.
   (b) For Fe 415 steel for all diameters 230 N/mm^2.

   [**Note:** As shells are thin structures in which corrosion of steel is very dangerous, it is advisable to use the modern TMT bars. *Tor steel should be avoided in shell constructions.*]

4. **Design constants for bending:** For design for bending we use the following formulae to find the depth of section and amount of steel required.

$$M = K(\sigma_{cl})(bd^2) \quad \text{and} \quad A_s = \frac{M}{f_s(jd)}$$

   (σ_{cb} = allowable bending stress given above)
   (a) For Fe 250 steel and *all grades of concrete*, we use $K = 0.174$ and $j = 0.87$.
   (b) For Fe 415 steel and *all grades of concrete*, we use $K = 0.131$ and $j = 0.90$.

   With Fe 415 and $f_{ck} = 20$; $M = 0.917\ bd^2$
   We may use SP. 16 for this purpose.

5. **Clear cover:** IS 2210-1988 specify a clear cover of not less than 13 mm or diameter of the bar whichever is more. (Generally, we provide a clear cover of 15 mm for slabs, 25 mm for beams, 40 mm for columns and 50 mm so for footings according to IS 456. Clause 26.4.)

6. **Concrete mix:** Modern specification always stipulates the grade of concrete to be used for shell construction. We specify a mix like 1 : 2 : 4 corresponding to grade M20 concrete for medium shells and 1 : 1.5 : 3 corresponding to grade M25 concrete for large shells. The maximum size of aggregates should be 20 mm. It should be 12 mm for places it is difficult to place the concrete. It is also better to work with a low water cement ratio using superplasticizer than use a very rich mix for satisfactory workability.

2.8 DETAILING OF STEEL

Concrete should easily flow to fill the full thickness of the shell at all points. This is very important. As shells are thin structures, it is very important to see that there is no congestion of steel in any part of the shell which will prevent free flow of concrete during the construction of the shell. *There have been cases of collapse of shells due to steel congestion which prevented free flow of concrete.* Crowded placement of steel should be avoided.

The steel reinforcements in each type of shell should be detailed according to standard practices for that particular type of structure. As continuity of concrete is very important in thin concrete, especially in compression, the construction joints should be properly planned. Similarly, removal of shutting of shells should be carefully carried out so that the shell action takes place for stability of the structure.

It may be pointed out that in the early days of use of cylindrical shells in India for large span warehouses, it was the practice to lay concrete in alternate layers to reduce shrinkage. Unless the joints are carefully laid, the various parts can act as very thin arches on removal of formwork. In fact, there was a case of failure in this type of construction

where the shell buckled as a thin arch. This practice to reduce shrinkage is not to be recommended as it is better to take care of shrinkage by extra reinforcement, if needed.

2.8.1 IS 2210–1988 General Recommendations

The following general recommendations are given in IS 2210 for detailing of steel in shells and folded plates.

The following rules apply to the general body of the shell:

The minimum steel diameter must be 8 m.

The maximum steel diameter should be 1/4 thickness of the shell or 16 mm, whichever is smaller. In the thickened portion, we may adopt the same rule as in slabs.

The maximum spacing of steel should be limited to five times the thickness of the shell and the area of the unreinforced panel should not exceed 15 times the square of the thickness of the shell (more details are given in Chapter 9). References 4 and 5 gives more details of the common types of shells used.

SUMMARY

This chapter gives an introduction to some of the commonly used shapes of shell roofs and their classification. It is summarized in Table 2.1. This chapter also gives a summary of the allowable stresses and loads to be used in the design of shell structures.

REVIEW QUESTIONS

1. Give a short account of the classification of shells.
2. Define Gauss curvature. What is its importance in shell design?
3. Give the sketches for at least eight types of reinforced concrete shells and three types of folded plates (refer IS) that are commonly used in practice.

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3

CLASSICAL METHOD OF ANALYSIS OF SHELLS

3.1 INTRODUCTION

There are many ways to study the theory and design of shell structures. A purely mathematical approach is difficult. Simplified mathematical study combined with a conceptual approach is the one most suited for beginners like students. Hence, we first take up the membrane theory which deals with the theory of shells whose bending rigidity can be neglected. This theory is applicable to a wide variety of shells. We then examine the shortcomings of this theory in the case of practical shell construction, and hence the need for the bending theory. Each type of shell gives a different expression for the membrane and bending theories. Thus, we will have different expressions for domes, cylinders, conical shells, hyperbolic paraboloid, etc. In this introductory chapter, we examine briefly the general mathematical principles of analysis which are the same for all types of shells.

3.2 STRESS RESULTANTS IN A SHELL

Shells are different from slabs. We have shells like an egg shell which is closed in shape or a sea shell which is not closed as in a roof. We will consider only thin shells which are used as roofs. Other types of thin shells and thick shells as used for industrial purposes and in atomic reactors are also important. They are very much different from these shells used in roof construction and we will not deal with them.

In general, there can be ten (which can be reduced to eight) stress resultants acting on a shell. These can be classified into the tangential group and normal group as shown in Figure 3.1.

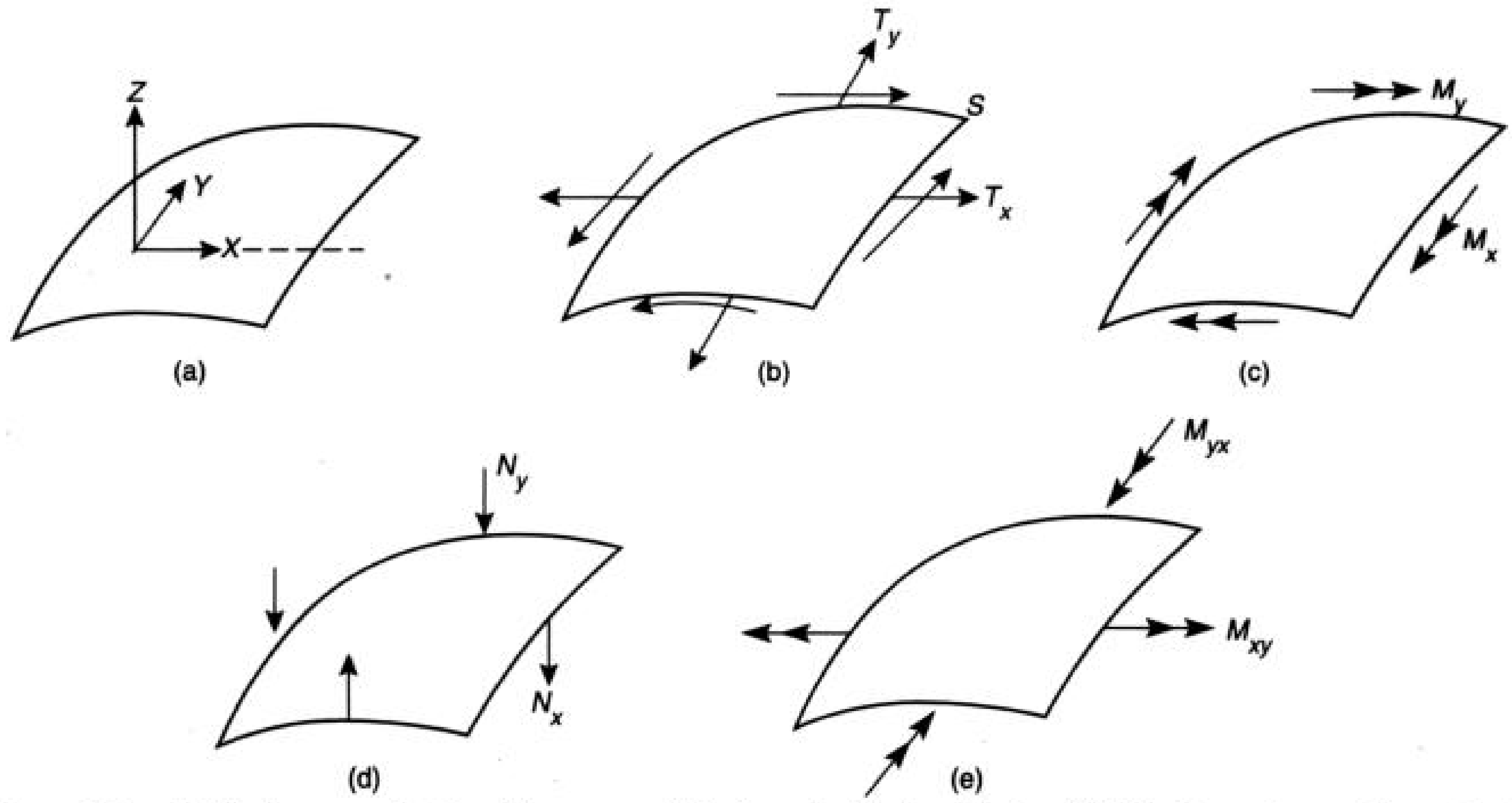

Figure 3.1 Eight stress resultants acting on a shell element: (a) Coordinates, (b) Membrane terces (3 numbers), (c) Bending moments (2 numbers), (d) Normal shears (2 numbers), and (e) Twisting moment (one number) (Total 8 forces).

T_x, T_y, S_{xy}, S_{yx}: Tangential group (4 nos.) reduces to 3 as $S_{xy} = S_{yx}$ [Two direct stresses and one shear, three in number.]

$\left.\begin{matrix} M_x, Q_x, M_{xy} \\ M_y, Q_y, M_{yx} \end{matrix}\right\}$: Normal group (6 nos.) reduces to 5 as $M_{xy} = M_{yx}$ [The five components are two moments, two normal shears and one torsion.]

This reduces the total number of indeterminates to eight in number.

[Notes:

1. The tangential (N) group is direct stresses and M group, the bending group. M_{xy} is the twist group. Generally, tension is taken as +ve and compression as –ve. (But the opposite signs are also used when the forces are mainly compression as in spherical domes.)
2. M_x increases to $M_x + \Delta M_x$ along X-axis. Similarly, M_y along Y-axis.
3. We may use the right hand screw rule (with palm facing up) for representing the direction of moments as shown in Figure 3.1. With the first finger moving forward, the movement of the right hand thumb shows the direction of the moment. In shells, moments producing tension in the inner fibres is considered positive.
4. Shears are considered positive when they create tension in the direction of increasing the value of x.]

3.3 STRUCTURAL ANALYSIS

Case 1: In the membrane analysis, we take that the bending and twisting moments are zero. Thus, in the tangential group, there will be only three unknown *stress resultants* T_x,

T_y and S. These can be determined by the three conditions of equilibrium of forces in the three directions. This is the membrane theory.

For example, in the membrane analysis of a cylindrical shell (Figure 3.1) we have to find T_x, T_y and S. We first sum up all the forces in the x (longitudinal) direction, then in ϕ (transverse) direction and also in the radial direction (the loads always act in the vertical direction).

From these three conditions we can solve for T_x, T_y and S. This is further explained in Chapter 5. Using stress-strain relationship, we can also *derive the deformations due to these membrane forces in the various directions.*

Case 2: In the bending analysis, if we take the middle surface as reference, the bending stress resultants acting on the shell can be reduced to five stress resultants. This reduces the total stress resultants to 3 + 5 = 8. This simplification will lead the general bending theory of shells to an eight degree partial differential equation with eight unknowns. This bending theory of shells is given in the next section.

3.4 BENDING ANALYSIS OF SHELLS

Edge disturbances in shells produce bending. In bending theory of shells, in addition to simple statics, we have to use deformation conditions also to solve the problems. The quantities we have in the case of a cylindrical shell are as follows [1]:

1. **Strains:** ε_x, ε_y—direct strain (2 nos.); γ—shear strain (1 no.)
2. **Displacements:** u—in longitudinal direction; v—in transverse direction; w—in radial direction
3. We also have K_x, K_y—curvatures (2 nos.) K_t—twist (1 no.).

We then proceed as follows for general bending analysis of shells:

1. Write down the force equilibrium equations.
2. Write down the stress-strain relations.
3. Write down the strain-displacement equations.
4. Write down the stress-displacement equations.

The resultant simplified equations after the above steps will give three simultaneous partial differential equations in u, v and w. By successive differentiation and elimination, we reduce the three partial differential equations to a single 8th order partial differential equation with only one unknown displacement, say, w. The resultant equation for *circular cylindrical shells* is known as the Donnel–Karman Jenkin (DKJ) equation. If we take $\mu = 0$ and displacement w as variable, the equation can be represented as follows [3]:

$$\beta\left[\frac{\partial^8 w}{\partial z^8} + 4\frac{\partial^8 w}{\partial z^6 \partial \phi^2} + 6\frac{\partial^8 w}{\partial z^4 \partial \phi^4} + 4\frac{\partial^8 w}{\partial z^2 \partial \phi^6} + \frac{\partial^8 w}{\partial \phi^8}\right] + \frac{\partial^4 w}{\partial z^4} = 0$$

As in the case of membrane analysis, in bending analysis also we can derive not only the stress resultants but also the expressions for the deformations of the shell.

3.5 DEFORMATIONS

In many cases of analysis of statically determinate structures, deformations are not considered as important. But in statically indeterminate system, no structural problem will be completely solved unless one has determined the corresponding deformations also at important points of the structure. Both compatibility of stresses and compatibility of deformations at junctions of elements to which we cut the structure have to be achieved.

Hence, in the highly indeterminate shells when using the membrane analysis for shells, we are interested not only in the membrane forces but also in calculating the membrane displacements. Formulas can be derived for these deformations also. Similarly, we should be able to calculate the displacements produced by other bending forces also in the shell.

We will see in Chapter 8 on analysis of cylindrical shells with edge beams that, it is through both the compatibility of stresses and deformations that we analyze shells with edge beams.

The pure mathematical approach, for example, to analyze a cylindrical shell will be to treat the structure as a whole as in References [1] to [3]. On the other hand, we may also analyze the indeterminate structure by first making it determinate, (under the action of membrane forces only), find the stresses and deformations at the edges. Then we calculate the additional stress resultants that are formed by making the stresses and deformations at the boundaries of the shell compatible. The sum of the membrane forces and the stress resultants obtained by making the edge deformations compatible will be the required stress resultants for the design. This method is followed in References [4] and [5]. We use this second approach in our study of circular cylindrical shells. ASCE Manual No. 31 gives us the tables for such an analysis and we will use that publication for our references.

It is also good to remember that the edge disturbances in shells of positive Gaussian curvature (like spherical domes) die down fast and that in shells with negative Gaussian curvature, the edge disturbances extend to more lengths into the shell from their edges. This will give an empirical approach to the design of shells telling us where we should be careful and where we can make approximations.

SUMMARY

This chapter deals with the classical method of analysis of shell roof structures. Both membrane and bending theories are necessary for a complete analysis of the shell. References [1] to [7] deal with the analysis of R.C. roof shells in great detail.

REVIEW QUESTIONS

1. Distinguish between "membrane forces" and "bending forces" acting on a shell.

2. How do you find a solution for the membrane forces? Indicate how you derive the differential equation for solving stresses and displacements in a shell. Indicate the nature of the differential equation with reference to a circular cylindrical shell.
3. Explain how equations for displacement of the shell for membrane analysis are important in the final analysis of the shell.
4. What is the difference between the membrane analysis and the bending analysis of shells? How many forces are to be solved in the first and how many in the second?

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4

DESIGN OF SPHERICAL DOMES AND CONICAL ROOFS

4.1 INTRODUCTION

Domes are shells of revolution. They can be of many types and shapes. Many of the old domes were ribbed domes built from masonry. They are very different from the modern R.C. domes. Appendix A gives some of the details of the construction of a few of these ancient domes. With reinforced concrete, we can design and build domes with much more ease.

Figure 4.1 shows a few of the different shapes of reinforced concrete domes that are generally used. First, we have the simple domes. An *Ogival dome* shown in Figure 4.1(a) has the shape of a Gothic arch. *Polygonal domes* are made of sections separated by ribs

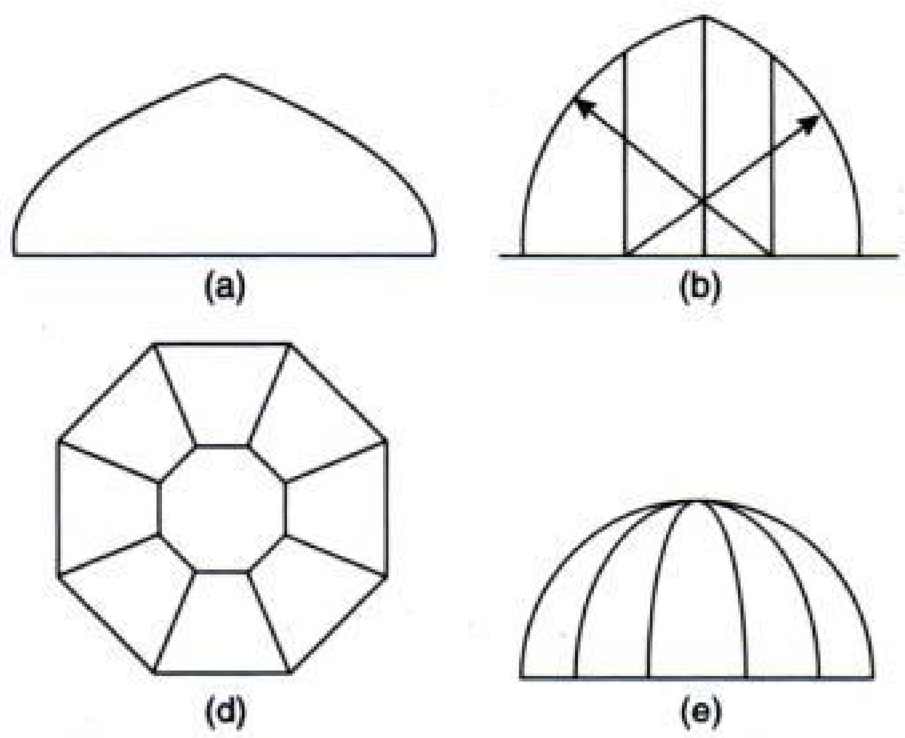


Figure 4.1 Different shapes of reinforced concrete domes, (a) Ogival dome, (b) Conoidal dome (c), (d) Elevation and plan of regular polygonal dome, and (e)(f) Elevation and plan of polygonal dome with circular sectors.

(also called *hips*) with a foot ring at the bottom. As the transfer of forces is through the ribs and not as hoop tension, the restriction of central angle of 52° to avoid tension (that we will study later in this chapter) does not apply. *Conoidal domes* are formed by the rotation of circular arcs not about the vertical axis but with axis parallel to it and some distance away from the central axis as shown in Figure 4.1(b).

Simple domes are surfaces of revolution formed by a simple curve about its vertical axis. The curve can be a circle (producing a *spherical dome*), an ellipse (producing an *elliptical dome*), and a hyperbola (producing a *hyperbolic dome*). *Conical domes* are formed by an inclined straightline fixed at one point and moving over a circle at the other end.

We will restrict our study in this chapter to simple spherical domes and conical domes. Conical domes are commonly used as roofs, especially when they have to support a heavy load at the top. Most of the domes built over monumental buildings are polygonal domes as shown in Figure 4.1. This reduces hoop tension considerably.

We must be aware that as domes are shells of positive Gaussian curvature, the edge disturbances do not travel large distances into the shell from the edges (Section 2.2.3). Hence, the membrane theory can be used for reasonable design of a large part of the shell. The edges should be thickened and specially reinforced to take care of the bending due to edge disturbances from membrane action. References [1] to [5] given at the end of this chapter can be used for a detailed study of the subject.

[Notes:

1. We should remember that angle ϕ in domes is measured from the crown. This is different from cylindrical shells where ϕ is measured from the edges.
2. In this chapter, we have taken **compression as positive and tension as negative** as is used in many standard textbooks for domes. In chapters on other shells, we have taken compression as negative. The signs used are indicated in each chapter.]

4.2 SPHERICAL DOMES

4.2.1 Planning of Spherical Domes

The following properties of spherical domes are important in their planning. When we consider domes under dead load, semicircular domes with full 90° half central angle will have no incompatibility with the reaction from the support it rests on. There is no ring tension. However, below the central angle of about 52°, it will experience hoop tension. On the other hand, segmental domes with less than 52° central angle will be fully in compression (meridinal and hoop compression), but the ends will require ring beams to take the horizontal component of the meridinal compression. (This is further explained under Section 4.2.3.)

Similarly, if we have a circular dome with an opening at its apex, there should be a ring beam at the top to balance the horizontal compression of the meridinal stress at the top (see Section 4.2.5).

It is also important that the shell should be so designed that concreting is easy. It is very difficult to lay concrete on any slopes greater than 30°–40° with the

horizontal without back formwork. Hence, we always plan for a half central angle equal to or less than 40°. With higher rise, the stresses will be low, but the slopes will be large.

For a 40° slope, the rise will be 1/5.49 span.
For a 35° slope, the rise will be 1/6.34 span.
For a 30° slope, the rise will be 1/7.46 span.

Accordingly, a rise-span ratio of 1/5–1/7 is usually used in practice. Even a rise of 1/8 span gives easy concreting and can give low stresses. The minimum thickness specified by IS 2210 is 50 mm but for practical purposes, we should adopt at least 75–100 mm.

Spherical domes up to a span of 60 metres can be built with R.C. ring beams. Larger domes will require prestressing of ring beams. Large monumental domes over important buildings are usually built as ribbed or polygonal domes with ornamentations and not as simple domes. Simple R.C. domes are commonly used for commercial purposes such as cover for water tanks.

4.2.2 Membrane Analysis

In shell design, dead load is taken as acting along the curved length of the shell, and live load is taken as acting on the plan area only. It can be shown mathematically for a uniformly distributed dead load that the forces developed in a spherical dome *under symmetric loading* will be only the meridinal stress N_ϕ (along the meridians or longitudes) and the hoop stress along the latitude or horizontal bands as shown in Figure 4.2. As uniform dead load on the shell will produce the same N_θ along the latitude, we can infer that the shear stress in such a surface will be zero under *such uniform loading*. Hence, we assume that the only two stresses that act on the main body of the spherical shell are the following. (In spherical shells, the angle ϕ is measured from the crown. We will see in Chapter 5 dealing with cylindrical shells that the angle ϕ is measured from the edge or springing.)

N_ϕ—Meridinal stress
N_θ—Hoop stress

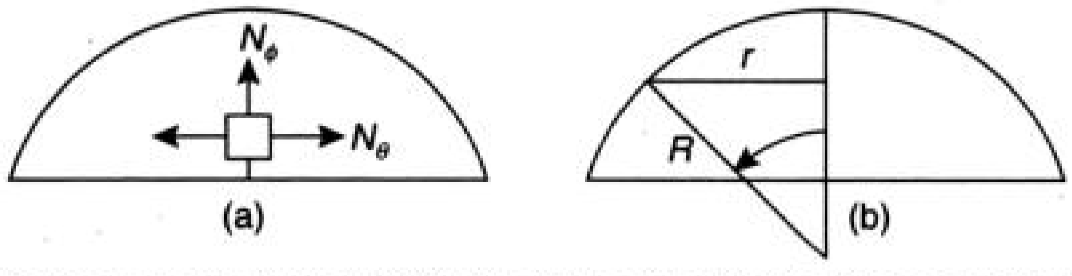


Figure 4.2 Membrane forces in a spherical shell: (a) Directions of membrane forces, and (b) Symbol R and r (different from cylindrical shells).

There can be a number of cases of shell analysis for various loadings such as:

1. Uniform dead load (DL) on a full shell surface with a central load on the apex
2. Uniform line loading over surface on top of a shell with skylight
3. Uniform live load assumed constant over the projected area (plan area) of shell surfaces
4. Shell with thickening of the edges and consequent variation of dead load

[**Notes:** As the slopes are generally below 40°, the effect of winds will be suction. Hence, the wind load analysis is carried out only for large and important shells only.]

In this chapter, we deal only with the first two cases. It is a common practice to adjust live load and even wind load to be considered as an equivalent dead load. But we must remember that non-uniform loading can produce shear stresses also. Wind loads on surfaces which are horizontal or inclined up to 40° to the horizontal (depending on height/width ratio) can produce suction, which is usually neglected as it reduces the stresses in the shell.

4.2.3 Case 1: Membrane Analysis for Uniform Dead Load and a Central Crown Load

Notes:

1. As we try to make most of the forces in domes as compression, most literature n domes **take compression as positive and tension as negative**. (This is different from the convention used in the analysis of cylindrical shells as given in Chapter 5.)

   (a) **Derivation of N_ϕ at ϕ with dead load w and a concentrated load W at the crown (Note ϕ is measured from the crown):**

   From statics, the vertical gravity load should be equal to the vertical component of N_ϕ.
   Surface area (of a dome of radius r and of height h) = $A = 2\pi rh$,
   where $h = R(1 - \cos\phi)$
   and $r = R \sin\phi$.
   Let w = Weight per unit area
   W = Concentrated load at crown
   Surface area of a dome = $2\pi R$ (height)
   Height 'h' at ϕ from crown = $R(1 - \cos\phi)$
   (Half span of dome = $r = R \sin\phi$)
   Consider a circle at angle ϕ from the crown.
   From Figure 4.4, resolving N_ϕ vertically and horizontally, with r = radius of circle at ϕ,

   $$N_\phi \sin\phi(2\pi R \sin\phi) = \text{Vertical load from top} = w(2\pi R) \times R \times (1 - \cos\phi) + W$$

   As $\sin^2\phi = (1 - \cos^2\phi)$, we get

   $$N_\phi = \frac{wR(1-\cos\phi)}{\sin^2\phi} + \frac{W}{2\pi R\sin^2\phi} \tag{4.1}$$

   or

   $$N_\phi = \frac{wR}{1+\cos\phi} + \frac{W}{2\pi R\sin^2\phi} \tag{4.1a}$$

   This is the *meridinal force per unit length* and its value increases with ϕ values (depth). It is always a compressive force.

   (b) **Value of N_θ:**

   Now, the component N_ϕ in the radial direction acting inwards and also the inward component of the dead weight produce N_θ. By equating forces in the radial direction, we can arrive at the following formula:

   $$N_\theta = wR\cos\phi - N_\phi \tag{4.2a}$$

or $$N_\theta = wR\cos\phi - \left[\frac{wR}{1+\cos\phi} + \frac{W}{2\pi R\sin^2\phi}\right] \tag{4.2b}$$

This is the *hoop force per unit length*, and the value of function tends to decrease with ϕ (depth) and will change in sign at a given value of ϕ.

At $\phi = 0$, we have

$$N_\phi = N_\theta = \frac{wR}{2} \text{ (with } W = 0)$$

Nature of Variation of N_ϕ and N_θ

All designers should have a clear idea of the following aspects.

The variation of N_ϕ and N_θ with uniform loading along the surface of shell is given in Figure 4.3.

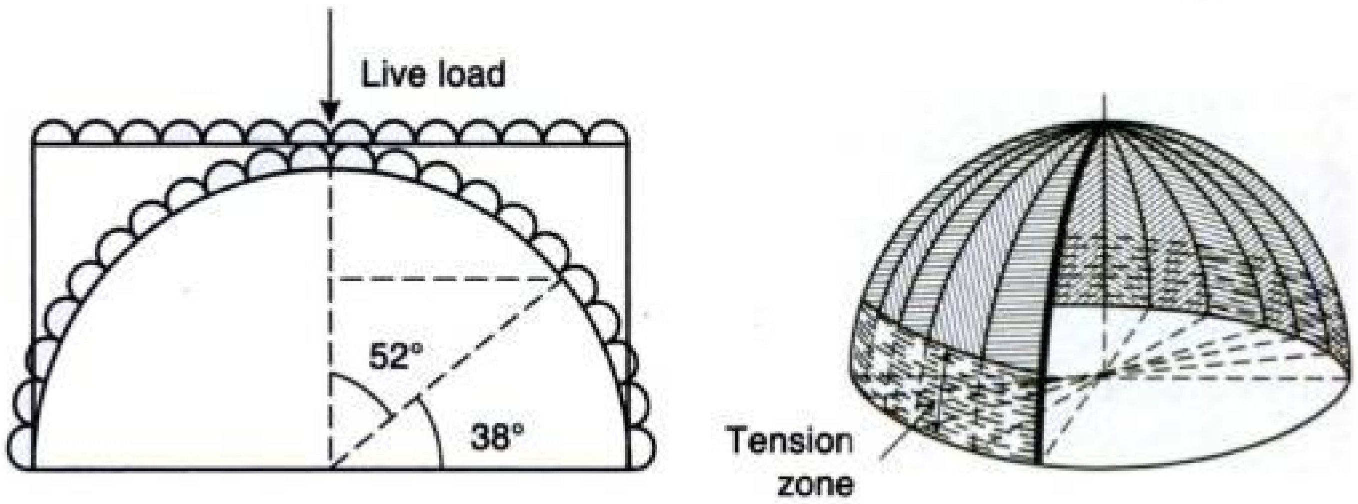


Figure 4.3 Distribution of dead and live loads.

The following five important features can be deduced from the formula for N_ϕ and N_ϕ:

1. Both N_ϕ and N_θ due to self load only are = $wR/2$ at $\phi = 0$.
2. The value of N_ϕ slowly increases with the value of ϕ.
3. The value of N_θ decreases with ϕ and *without any apex load* W and it becomes zero when

$$\cos\phi - \frac{1}{1+\cos\phi} = 0 \quad \text{or} \quad \phi = 51°.48'$$

Hence, beyond approximately 52° central half angle, M_θ is *tension when only self load* is acting as shown in Figure 4.4. We can also imagine how, by keeping the dome light in weight in the apex or higher portions, we can reduce the thrust and also lower the height of the tension angle. This principle was one of the methods of construction of old circular masonry domes (see Appendix A).

4. When $\phi = 90°$, the direction of N_ϕ is normal to the support and hence there is no horizontal tension.
5. When $\phi < 90°$, there will be vertical and horizontal components for N_ϕ as shown in Figure 4.4.

(It is interesting to compare the above stress distribution with that of a conical roof supported on its edges as described under conical domes)

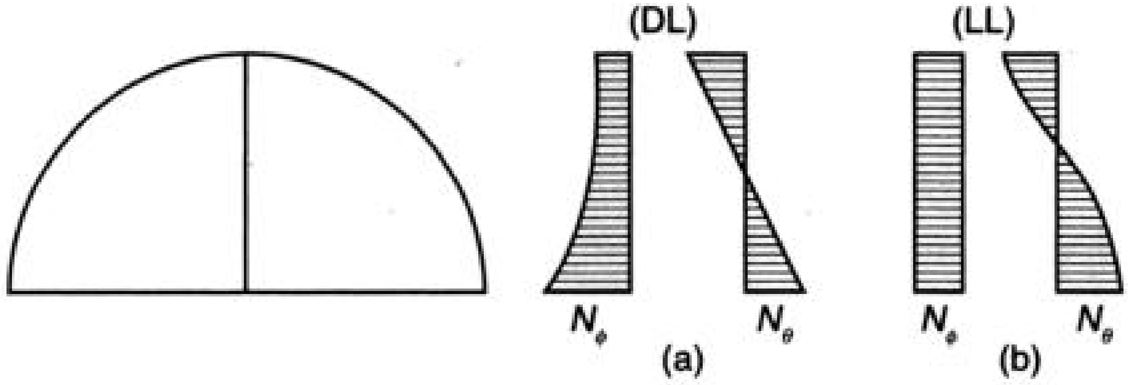


Figure 4.4 Distribution of membrane forces in a spherical shell due to dead and live loads: (a) N_ϕ and N_θ due to dead load, and (b) N_ϕ and N_θ due to live load.

4.2.4 Analysis for Live Load

Live load is taken on the plan area as shown in Figure 4.3. Usually, as live loads are small, an equivalent dead load can be taken in lieu of live load. It is, however, interesting to study the effect of live load. We can derive the following expression for N_ϕ and N_θ for live loads:

$$N_\phi = \frac{wR}{2} \text{ (constant compression)} \tag{4.3}$$

$$N_\theta = \frac{wR}{2}\cos 2\phi \text{ (compression changes to tension at } \phi = 45°) \tag{4.4}$$

This is also indicated in Figure 4.4.

4.2.5 Analysis of Domes with Skylight

Many domes are not closed at the vertex but are provided with a circular skylight, lantern, ventilation device, etc. as shown in Figure 4.5. Let us assume the following:

We take the load exerted by skylight = P per unit length

The half angle subtended by skylight = ϕ_0

Radius of shell = R

Weight of dome above $\phi° = w \times 2\pi R \times R(1 - \cos\phi) = 2\pi R^2 (1 - \cos\phi)$

At angle ϕ from Equation (4.1) and Figure 4.6, we get

$$N_\phi = \text{Due to [DL on full area – DL on skylight area] + Due to P}$$

As we derived Equation (4.1), considering vertical loads,

$$N_\phi(2\pi R \sin^2\phi) = w(2\pi R^2)[(1 - \cos\phi) - (1 - \cos\phi_0)] + P \times (2\pi R \sin\phi_0)$$

$$N_\phi = \frac{wR(\cos\phi_0 - \cos\phi)}{\sin^2\phi} + \frac{P\sin\phi_0}{\sin^2\phi} \tag{4.5}$$

and $$N_\theta = wR\cos\phi - N_\phi \text{ [as in Equation (4.2a)]} \tag{4.6}$$

[**Note:** When $\phi = \phi_0$, we get $N_\phi = P/\sin\phi_0$ which is true by simple statics as shown in Figure 4.5. The maximum value of N_ϕ due to P occurs at upper edge.]

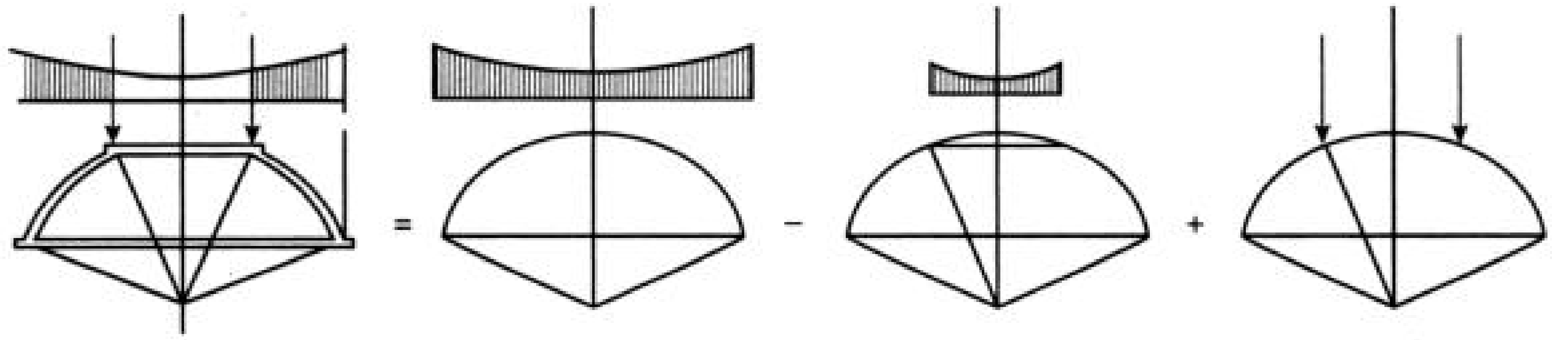


Figure 4.5 Analysis of domes with skylight.

4.2.6 Design of Ring Beams (Edge Member)

As we have seen in Section 4.2.4, if N_ϕ is not vertical (as in a segmental dome), we have to provide for the horizontal and vertical components of N_ϕ as shown in Figure 4.6. Ring beams are provided at the base in all cases and also at the skylight level for domes with skylight.

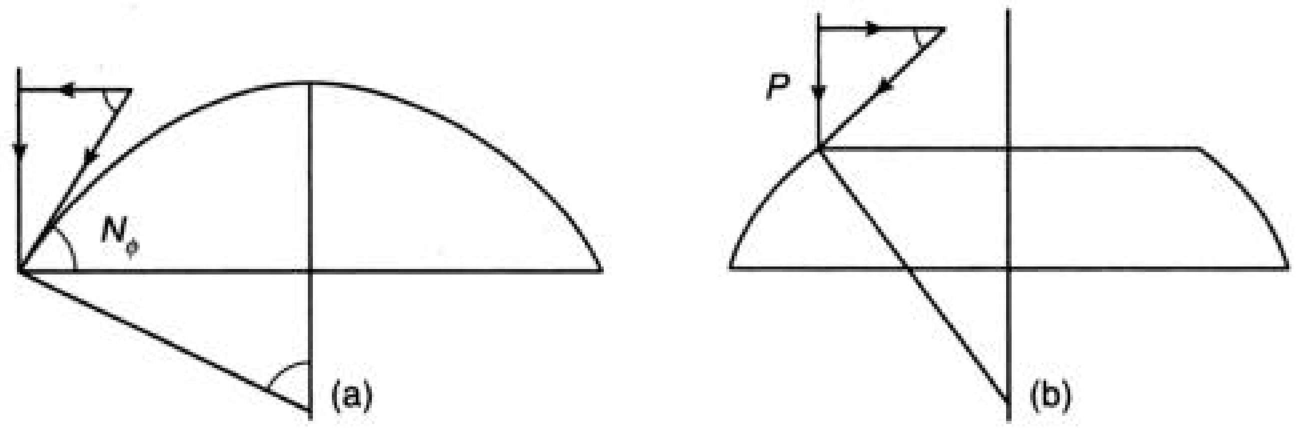


Figure 4.6 Design of ring beams of a dome: (a) Bottom ring beam, and (b) Top ring beam of skylight.

Design of Ring Beam at Base

From Figure 4.6, we have *component of* $N_\phi = \bar{t}$ as follows (symbol t is used as thickness of shell):

$$\bar{t} = N_\phi \cos \phi \qquad (4.7)$$

where ϕ is the half base angle. This tension acts radially along the full base circle.

$$\text{Total hoop tension on ring beam} = N_{\phi(\max)} \times \cos \phi \times R \sin \phi \qquad (4.7a)$$

where $R \sin \phi = B/2$ = One half span of shell.

[We may also derive $\bar{t}$ from V equal to the *total load on dome per* m *length of circumference of ring beam* as in Example 4.1.)

V for a shell with only self weight w/m^2

$$V = \frac{\text{Total load}}{2\pi \times \text{half span}} = \frac{2\pi Rhw}{2\pi r} = \left(\frac{R}{r}\right) hw \qquad (4.8)$$

$$\bar{t} = V \cos \phi \qquad (4.8a)$$

where V is the total load of the shell per meter length of circumference near ring beam.

$$\text{Total tension in ring beam } T = (\bar{t} \times \text{span}/2) \qquad (4.8b)$$

As it is a tensile member, we compute its size as follows:

1. First of all, find the area of steel A_{st} for tension assuming allowable tension,

$$A_{st} = T/f_s \tag{4.9}$$

2. The area of concrete should be such that the stress in concrete should not be high,

$$\frac{T}{A_c + (m-1)A_t} \le f_{ct} \text{ (Allowable tension in concrete, say, } f_{ck}/10))$$

$$A_c = \frac{T}{f_{ct}} - (m-1)A_{st} \tag{4.10}$$

3. All dimensions of the ring beam should be not less than twice the thickness of the shell.

Design of Ring Beam of Skylight

Ring beam of skylight distributes the load from the skylight to the shell. Hence, it will be in *ring compression*. Taking as vertical load per metre length,

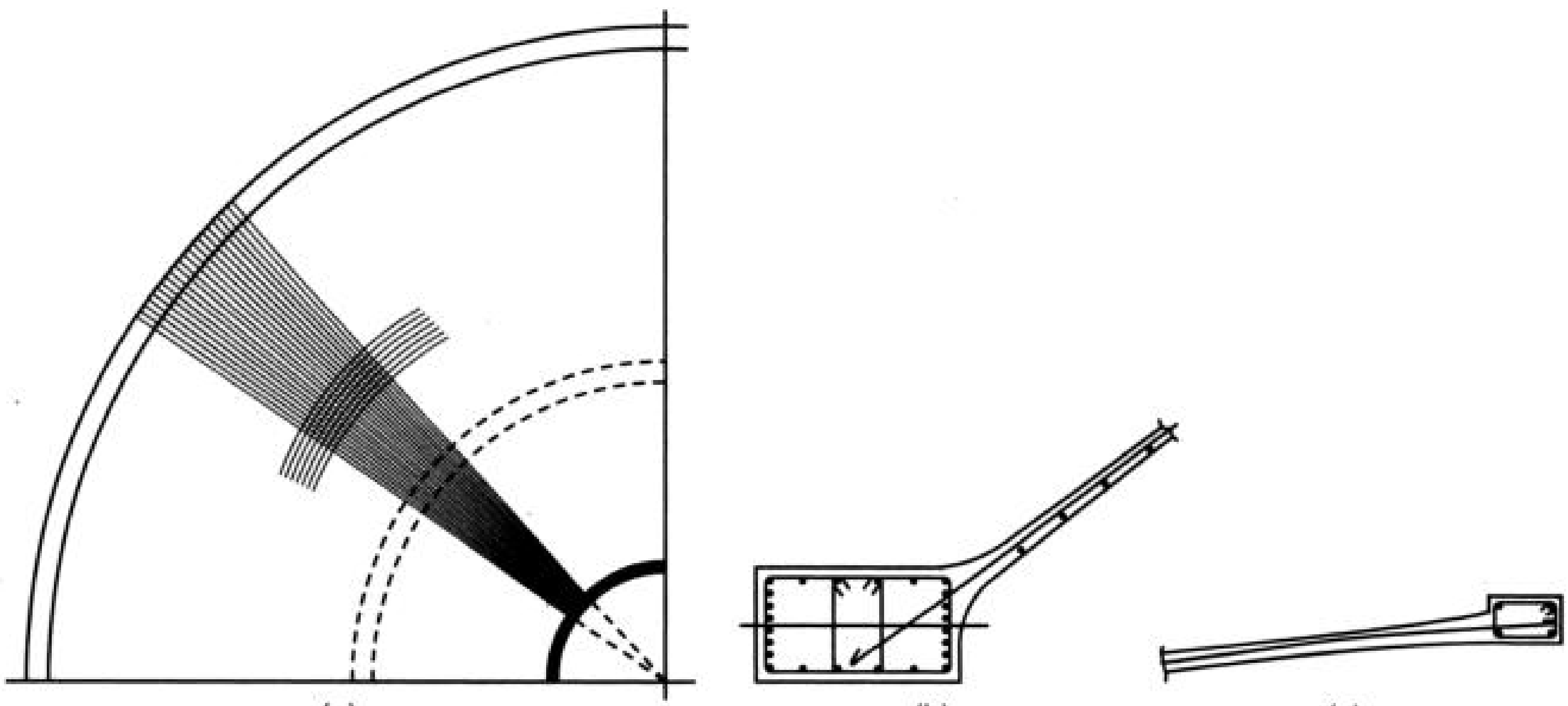
$$\text{Value of } N_\phi = \frac{P}{\sin \phi_0}$$

and the horizontal thrust $H = P \cot \phi_0$.

Hence, compression in top ring beam = (Radius of ring of skylight)($P \cot \phi_0$) (4.11)

4.2.7 Design for Shear between Bottom Ring Beam and Dome

The vertical component of the thrust between the shell and ring beams at the bottom produces shear between the junction of the shell and the ring beam YY as shown in Figure 4.7. This shear at the junction should be checked and the extra tension steel provided should enable the concrete to carry the shear without vertical stirrups. This shear will also be the total weight of dome transferred at the junction between the ring beam and the dome.



(a) (b) (c)

Figure 4.7 Detailing of steel in spherical dome with sky light: (a) Plan, (b) Bottom ring beam, and (c) Top ring beam.

4.2.8 Detailing of Steel

1. *Main steel on shell:* Generally, the meridinal steel in the shell is placed at the bottom of the surface and the circumferential steel (hoop steel) on the top with equal cover on top and bottom. (The reverse is also allowable.) In most cases, the stresses will be compression and theoretically no steel will be needed. The minimum steel for water tank domes *is usually specified as* 0.3% for mild steel and 0.2–0.24% for HYSD bars both ways. As the meridinal steel will crowd towards the crown, alternate bars are discontinued at one half distances. Thus, start with the number of bars as multiples of four at the outer edge, cut off alternate bars at 1/2 the surface length from the centre (and provide development length), and again cut off alternate bars at 1/4 the same surface length. It will be difficult to continue all steel through the centre, so provide a rectangular grid at the centre (as in the case of conical shell shown in Figure Example 4.5).
2. *Detailing the ring beam at the bottom:* As this ring beam is in tension, provide the necessary steel and concrete size of member. Usually, as bending will be present at the junction because of the edge member, we increase the thickness of the steel at the junction. As additional steel, we provide extra steel (the same amount as meridinal steel) at the top and bottom of the shell (to take care of bending) and extend it into the shell for a distance equal to at least 10 times the thickness of the shell as shown in Figure 4.7.
3. *Detailing of steel in top ring beam:* This ring beam is in compression. The size of this ring beam should be not less than twice the thickness of the shell and detailed as shown in Figure 4.7. Steel equal to at least 0.6% of the area is provided as minimum steel.

4.3 CONICAL SHELLS

Conical shells are used as roofs, bottom part of intze type of water tanks, bottom part of bunkers, etc. As roofs, they can be used in the following two ways (Figure 4.8):

- *Conical dome roof* supported at the base with a ring beam at the base
- *Conical umbrella roof* supported at the apex

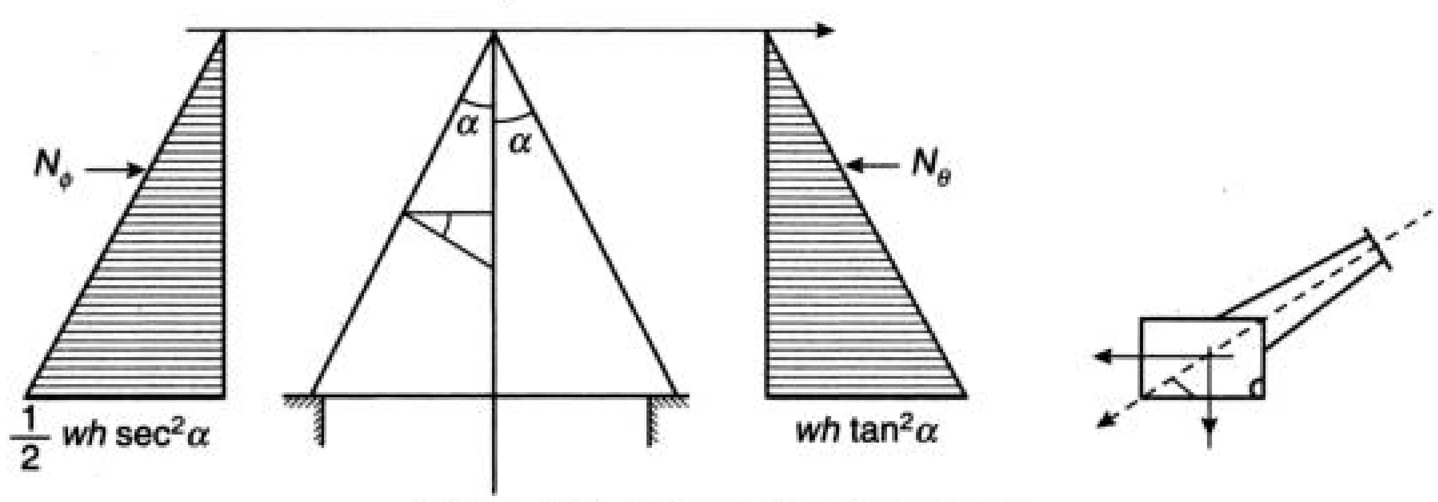


Figure 4.8 Analysis of conical domes.

As explained in Appendix A, the heavy cross of St. Paul's Cathedral in London is supported by a conical roof inside the circular dome. In this chapter, we examine the forces that act on these two types of shell roofs.

4.3.1 Conical Dome Roof with Ring Beams

Conical dome roofs are the roofs supported at the base as shown in Figure 4.8. It is always provided with a ring beam. The main parts of the shell are the shell proper and the ring beam. The membrane forces acting on the shell are the meridinal N_ϕ forces and the hoops N_θ forces. Both are compressive and zero at the vertex and increase linearly with the depth as shown Figure 4.8.

We take the origin at the vertex and measure downwards. Let the central axis be Z axis and the depth of the *point from vertex defined as* z. Let half the central angle be α. We generally keep the angle of the core to the horizontal to be such that we can concrete the shell without difficulty (to less than 35°). Consider a point at depth z. Let the superload on shell be w/m^2.

1. Value of N_ϕ:

   Load on surface $= w \times (\pi r L) = w\pi(z \tan \alpha)(z \sec \alpha) = w\pi z^2 \tan \alpha \sec \alpha$

   N_ϕ acts along the slope equating its vertical component to vertical load on shell.

$$N_\phi\,(2\pi z \tan \alpha \cos \alpha) = w\pi z^2 \tan \alpha \sec \alpha$$

$$N_\phi = \frac{wz \sec^2 \alpha}{2} \quad \text{for any value of } z \text{ from vertex} \tag{4.12}$$

2. Value of N_θ:

$$2N_\theta = (\text{Normal pressure}) \times 2 \text{ radius of action}$$

   Component of weight normal to cone $= w \sin \alpha$

$$N_\theta = (w \sin \alpha) \times (\text{Radius normal to shell} = r_1)$$

$$N_\theta = w \sin \theta \times z \tan \alpha \sec \alpha = wz \tan^2\alpha \text{ at any value of } z \tag{4.13}$$

3. Tension due to horizontal component of compression at the base of depth h:

$$T = (\text{Horizontal component}) \times (\text{Radius})$$

$$T = (N_\phi \text{ at } h) \sin \alpha \times h \tan \alpha = -\frac{wh^2 \tan^2 \alpha \sec \alpha}{2} \tag{4.14}$$

The distribution of forces is shown in Figure 4.8. The stresses are compressive. This tension component at the base has to be taken up by using a ring beam as in the case of a dome.

4.3.2 Umbrella Roof

Conical shell can also be used as an umbrella roof supported at its vertex as shown in Figure 4.9. We take the origin at the vertex.

At a point z below the vertex, the value of tension N_ϕ will be due to the drag of the weight of the shell below the depth z, i.e. $(h - z)$. Where h is the height of the cone

$$2\pi r N_\phi \cos\alpha = w\pi(h^2 - z^2)\tan\alpha\sec\alpha$$

$$\text{Meridinal tension} = N_\phi = -\frac{w(h^2 - z^2)\sec^2\alpha}{2z} \tag{4.15}$$

The value of N_ϕ *in tension is zero* at $z = h$ (at the lower edge of the umbrella roof) and increases non-linearly and becomes infinite at $z = 0$ at the crown.

Similarly,

$$N_\theta = (\text{Normal component}) \times (\text{Radius } r_1)$$

$$N_\theta = (w \sin\alpha)(z\tan\alpha\sec\alpha) = wz\tan^2\alpha \text{ (hoop compression)} \tag{4.16}$$

It is zero at $z = 0$ and increases linearly to a maximum value at $z = h$.

The distribution of the stresses N_ϕ and N_θ will be as shown in Figure 4.9.

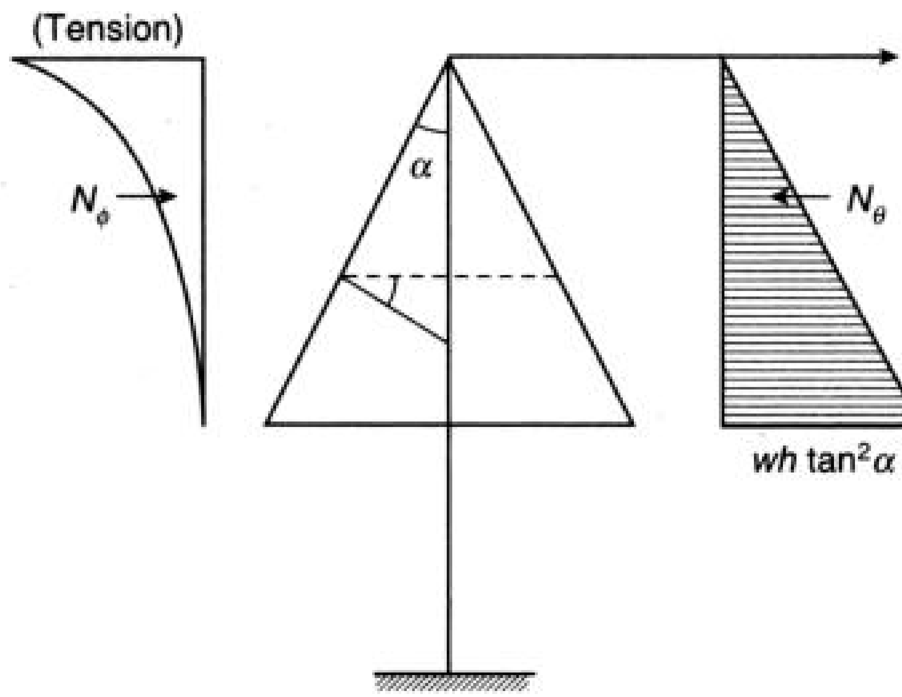


Figure 4.9 Analysis of conical umbrella roof.

SUMMARY

The design of circular domes, conical domes and umbrella roof shells are explained in this chapter. All these shells are subjected to meridinal and hoop forces. The distribution of forces in circular domes is shown in Figure 4.4 and that of conical shells in Figures 4.8 and 4.9. Segmented circular domes and conical domes must have ring beams to take care of tension. In the umbrella roof, the value of hoop stress is compressive but the meridinal stresses are tension. It is better to provide a small ring beam at the bottom of umbrella roofs for appearance and also better performance at ultimate load (overload). As the tension due to M_ϕ in domes with skylight at the vertex is infinite, the steel in that part should be properly detailed. Formulae for membrane forces in the various types of shells are summarized in Table 4.1.

TABLE 4.1 Formulae for Forces in Spherical and Conical Shells

1. **Spherical Dome**

   Dead load = w/m^2; Central load = W; Skylight canopy load = P/m (Tension force is indicated as –ve)

   (a) *Full dome:*

$$N_\phi = \frac{wR}{1+\cos\phi} + \frac{W}{2\pi R\sin^2\phi}\left[\text{First term also} = \frac{wR(1-\cos\phi)}{\sin^2\phi}\right] \quad (4.1)$$

$$N_\theta = wR\cos\phi - N_\phi \quad (4.2)$$

   (b) *Dome with skylight:*

$$N_\phi = \frac{wR(\cos\phi_0 - \cos\phi)}{\sin^2\phi} + \frac{P\sin\phi_0}{\sin^2\phi} \quad (4.5)$$

$$N_\theta = wR\cos\phi - N_\phi \quad (4.6)$$

$$\bar{t} = \frac{V}{\cot\phi} \quad (4.8)$$

   (where V is total load per metre of circumference near ring beam)

$$T = \bar{t}\times\left(\frac{\text{span}}{2}\right) \quad (4.8a)$$

2. **Conical Dome**

$$N_\phi = wz\sec^2\frac{\alpha}{2} \quad (4.12)$$

$$N_\theta = wz\tan^2\alpha \quad (4.13)$$

   Ring beam, $$T = -wh^2\tan^2\alpha\sec\frac{\alpha}{2}\text{(tension)} \quad (4.14)$$

3. **Umbrella Roof**

$$N_\phi = \frac{w(h^2 - z^2)\sec^2\alpha}{2z}\text{ (tension)} \quad (4.15)$$

$$N_\theta = wz\tan^2\alpha\text{ (compression)} \quad (4.16)$$

EXAMPLE 4.1 [Design of a simple spherical dome (segmental dome)]

A dome for a water tank is 12.5 m in span. Design the dome and ring beam. (Use elastic design with Fe 250 steel. Allowable steel stress = 140 N/mm^2) [5] [**Note:** R = Radius of dome and span is sometimes referred as $S = 2r$.]

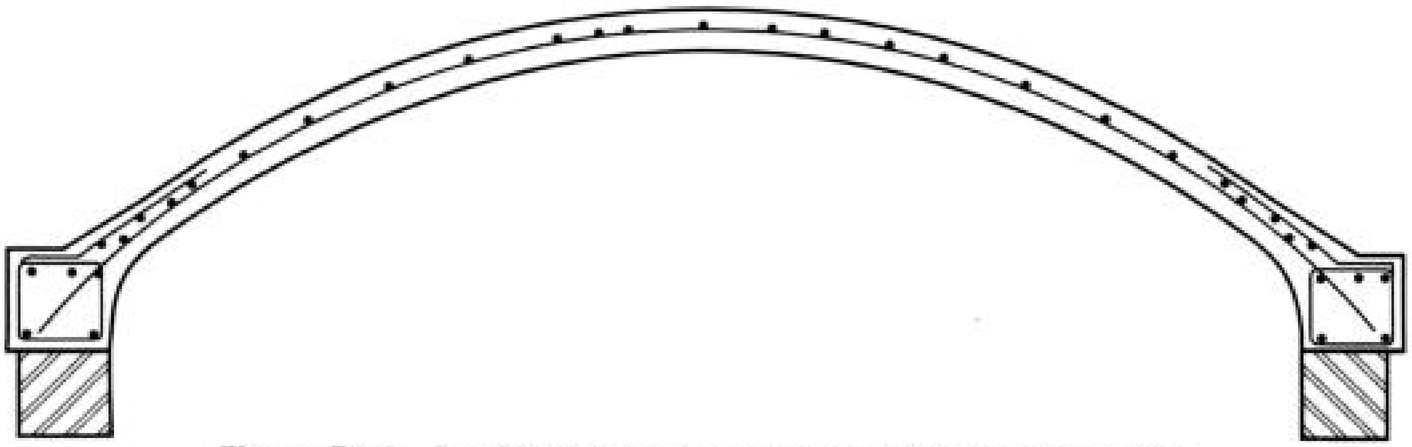


Figure E4.1 Detailing of reinforcements in the spherical dome.

Reference	Step	Calculations
	1.	*Find radius of dome = R. Let span = S*

Adopt a rise of 1/6 span, say, 2 m.

r = Span/2 = 6.25 m; rise h = 2 m; t = 0.15 m (150 mm)

$$\text{Radius of shell } R = \frac{(S/2)^2 + h^2}{2h} = \frac{(6.25)^2 + (2)^2}{2 \times 2} = 10.75 \text{ m}$$

$$\text{Total } \phi = \sin^{-1}\left(\frac{6.25}{10.75}\right) = 35°48'(35.8°) < 51°52'$$

[**Note:** As semicentral angle is 35.8°, all forces are compression.]

2. *Find load = w*

Dead weight of shell = 0.15 × 1 × 24 kN = 3.6 kN/m^2

Assume equivalent wind and other live loads

@1.5 kN/m^2 = 1.5 kN/m^2 (As = ϕ is nearly 40°)

Water proofing @ 0.9 kN/m^2 = 0.9 kN/m^2

Characteristic load = Total load = 6.0 kN/m^2

3. *Find stresses in the shell (elastic design)* N_ϕ and N_θ

$$N_\phi = \frac{wR}{1 + \cos\phi}$$

$$\frac{wR}{1 + \cos\phi} = \frac{6 \times 10.75}{2} = 32.25$$

Table E4.1 Calculation of N_ϕ and N_θ

[w = 6 kN/m^2; R = 10.75 m; **Use Eqs. (4.1) and (4.2)**]

Eq. (4.1)

Eq. (4.1)

1	2	3	4	5	6
ϕ =	$\frac{wR}{1+\cos\phi}$	$\frac{W}{2\pi R \sin^2\phi}$	N_θ=Col(2+3)	$wR\cos\phi$	N_ϕ=Col(3–4)
0	32.25	0	32.5	64.5	32.25
↓	*Enter values for other values of ϕ*				
35.8	35.6	0	35.6	52.3	16.7

Maximum compression due to N_ϕ = 35600/150 × 1000 = 0.24 N/mm^2

These stresses are low, and only nominal steel is necessary. We provide 0.12–0.3% steel both ways using high yield deformed bars.

$$\text{Assume } 0.2\%\ A_s = \frac{0.2}{100} \times 150 \times 1000 = 300 \text{ mm}^2/\text{m}$$

Provide 8 mm @ 15 cm (335 mm^2) as hoop and meridinal reinforcement.

Reference	*Step*	*Calculations*
	4.	*Check compression for buckling*
		σ_{cr} assuming $E = 25 \times 10^3$ N/mm^2
Chapter 19		$\sigma_{cr} = \dfrac{Et}{R\sqrt{3}}$ (see Chapter 19)
		$= \dfrac{25 \times 10^3 \times 150}{10750 \times \sqrt{3}} = 201$ N/mm^2
		With safety factor 4 = 50 N/mm^2.
		Hence very safe against buckling.
	5.	*Calculate tension T and design ring beam*
		Method 1: From N_ϕ of Table E.4.1 (Step 2)
		Hoop tension in ring beam = $(N_\phi \cos \phi)(1/2$ span$)$
		$\bar{t} = N_\phi \cos \phi = 35.6 \times \cos 35.8 = 28.8$ kN
		$T = (\bar{t} \times \text{span}/2) = 28.8 \times 6.25 = 180.5$ kN
		Method 2: From total load on dome [Eq. (4.8)]
Eq. 4.8		V per m $= \left(\dfrac{R}{r}\right) wh = \dfrac{10.75 \times 2 \times 6}{6.35} = 20.64$ kN/m
Eq. 4.8a		$\bar{t} = V \cot \phi = 28.8$ kN/m
		$T = (\bar{t} \times \text{span}/2) = 28.8 \times 6.25 = 180.5$ kN
		Design ring beam by elastic theory
		(a) Area of steel $= \dfrac{180500}{140} = 1290$ mm^2
		Provide 8 nos. 16 mm (1608 mm^2)
		(b) Area of concrete (Assume $\sigma_{tension} = 1.1$ N/mm$^2 = \sigma_{ct}$)
		$\sigma_{ct} = \dfrac{T}{A_c + (m-1)A_s} = \dfrac{179000}{A_c + (17 \times 1608)}$
		Assume $m = 18$ for A_s.
		$A_c = 127000$ mm^2 (Approx.)
		Use 500 × 300 mm ring beam.
	6.	*Design for shear between bottom ring beam and dome*
		Shear = Weight of shell = $(2\pi Rh) \times w/2\pi \times$ (half span)
		$= \dfrac{10.75 \times 2 \times 6}{625} = 20.64$ kN/m = 20640 N/m
		Steel required $= \dfrac{20640}{140} = \dfrac{147 \text{ mm}^2}{\text{m}}$
		Provide 8 mm @ 30 cm (gives 167 mm^2)

EXAMPLE 4.2 [Design of spherical segmental dome with a canopy, (see also Example 4.3)] Design a reinforced concrete dome resting on a circular 340 mm thick brick water tank, 11 m in diameter. Assume the dome is 100 mm thick. At the top of the dome, a circular canopy 1.75 m in diameter is provided for ventilation and its total weight is estimated as 20 kN. Design the reinforcements for the main dome and also the top and base ring beams. (Use Fe 415 steel [5].)

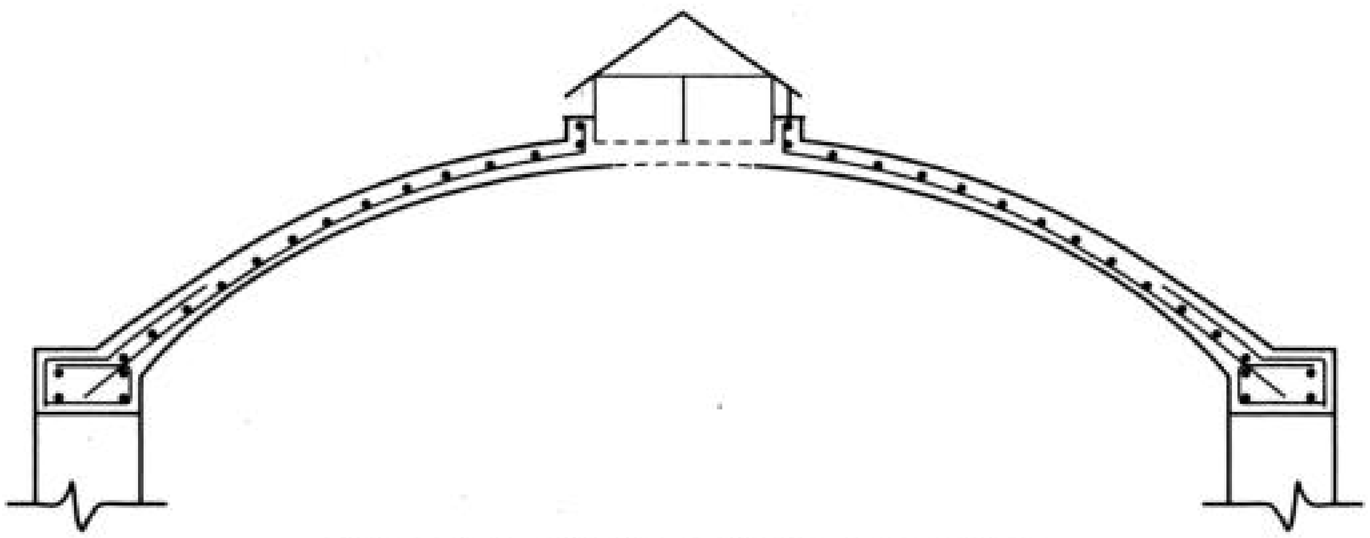


Figure E4.2 Detailing of dome with sky light.

Reference	*Step*	*Calculations*
	1.	*Adopt usual size for the dome*
		Adopt rise 1/5 span for the dome
		Span $S = 11.0 + 0.34 = 11.34$ m; $S/2 = r = 5.67$ m
		$h = 11.34/5 = 2.27$ m; let us also provide a rise h of 2 m.
		We can express radius also as,
Ex. 4.1		$$\frac{h}{2}+\frac{S^2}{8h}=\frac{2}{2}+\frac{(11.34)^2}{8\times 2}=9\text{ m}$$
		ϕ_s, half central angle; $\phi_s = \frac{S}{2R}=\frac{11.34}{18} = 0.63$. Hence $\phi_1 = 39°$
		Find ϕ_0 of opening where shell starts. Half angle = ϕ_0
		$$\sin\phi_0 = \frac{d}{2R}=\frac{11.75}{2\times 9} = 0.097.\ \phi_0 = 5.58°$$
		$$h_0 = R(1-\cos\theta_0) = 0.043\text{ m}$$
		Adopt thickness of shell = 100 mm
		Buckling compression for $f_{ck} = 25$ and $E = 2.5 \times 10^6$ N/mm^2
		$$f\text{ critical} = \frac{Et}{R\sqrt{3}} = \frac{2.5\times 10^4\times 100}{9000\times\sqrt{3}} = 160\text{ N/mm}^2$$
		Allowable comp. for M25 = 6 N/mm^2 $\left(\approx \frac{1}{4}f_{ck}\right)$

Reference	*Step*	*Calculations*
	2.	*Estimate loads*
		Dead weight = 0.1 × 25 = 2.5 kN/m^2
Sec. 2.7		Assume live load to be equivalent to dead load, = 1.5 kN/m^2
		Total UDL on shell = 4.0 kN/m^2 (1)
		Load P due to ventilator $= P = \dfrac{20}{\pi d} = \dfrac{20}{\pi \times 7.5}$ = 3.64 kN/m (2)
	3.	*Find N_ϕ and N_θ for different ϕ values*
		Estimation of N_ϕ and N_θ
		(w = 4 kN/m; P = 3.64 kN/m; R = 9 m; ϕ_0 = 5.58° and ϕ_k = 39°)
Eq. (4.3)		
Eq. (4.4)		

1	2	3	4	5	6
ϕ	$\dfrac{wR(\cos\phi_0 - \cos\phi)}{\sin^2\phi}$	$\dfrac{P\sin\phi_0}{\sin^2\phi}$	N_ϕ (2+3)	$wR\cos\phi$	N_θ (5 − 4)
5.58	0	37.5	37.5	35.8	−1.7
Enter values for other values of ϕ					
15	15.6	5.3	10.9	34.9	14.0*
39	19.8	0.9	20.7	28.0	7.3

		*Max. value of N_θ will be = 15.5 kN/m
	4.	*Design the shell for maximum meridinal and loop stresses*
		(a) Max meridinal compression = 37.5 kN/m
		Max. stress $= \dfrac{37.5 \times 1000}{1000 \times 100}$ = 0.37 N/mm^2 (less than allowable)
		Provide the minimum nominal steel of 0.12% to 0.3% of concrete area both ways
		$= \dfrac{0.12 \times 1000 \times 100}{100} = 120\ \text{mm}^2/\text{m}$
		8 mm at 270 gives 186 mm^2/m (nominal)
		(b) *Maximum hoop tension* = 15.5 kN/m
		$A_s = \dfrac{15.5 \times 1000}{\text{Allowable stress}}$ = per metre length
		(For 415, use allowable stress, $\dfrac{0.817 \times 415}{1.5}$ = 240 N/mm^2)
		$= \dfrac{15.5 \times 1000}{240} = 65\ \text{mm}^2/\text{m}$
		Minimum steel 120 mm^2/m (see above)
		Provide 8 mm @ 270 mm (nominal)

Reference	*Step*	*Calculations*
	5.	*Design at base ring beam* [We can use Eq. (4.7) as there is a canopy]
		(a) *Design steel*
Eq. (4.7)		Tension $T = N_\phi \cos \phi_K \times r = 20.7 \times \cos 39 \times 5.67 = 91$ kN
		Steel area $= \dfrac{91 \times 1000}{240} = 379\ \text{mm}^2$
		Provide 4 nos. 12 mm bars gives 452 mm^2 area of steel
		Add 8 mm stirrup at spacing 200 mm (d)
		(b) *Section of beam for limiting concrete stress*
		Assume $m = 9$ for cracking and cracking stress $= \dfrac{f_{ck}}{10} = 2\ \text{N/mm}^2$.
		Let the area of concrete required be A_c.
		$[A_c + (m - 1)A_s] \times 2 = 90.75 \times 1000(\text{N})$
		$A_c = 41{,}759\ \text{mm}^2$. Let breadth = 340 mm
		Provide 340 × 200 mm ring beam
	6.	*Design top compression ring for canopy*
		The ring beam distributes the load as a line load and also takes care of the bending stresses due to discontinuity. $r = 1.75/2 = 0.875$ m
Eq. (4.9)		Compression $H = P \cot \phi_0 \times (\text{radius}) = \dfrac{3.64 \times 0.875}{\tan 5.58} = 32.6$ kN
		As the shell is *100 mm thick*, we increase the depth to 200 mm and adopt 200 × 200 mm thickening.
		Stress $= \dfrac{32.6 \times 1000}{200 \times 200} = 0.8\ \text{N/mm}^2$
		Provide about 0.4% steel for compression
		$A_s = \dfrac{200 \times 300 \times 0.4}{100 \times 0.6} = 240\ \text{mm}^2$
		Provide 4 rods of 8 mm (201 mm^2) with 6 nn ties at 200 mm spacing as reinforcement.

EXAMPLE 4.3 (Design of a spherical dome with skylight)

Design a spherical dome with the following data [4] (Figure 4.7):

Radius of dome $R = 50$ m

Thickness of dome = 60 mm

Lower ring half angle $= \phi_K = 36°$

Lower ring radius $r = 50 \sin 36 = 29.39$ m

Upper ring half angle $= \phi_O = 6°$

Ring load of skylight = P = 8 kN/m
Assume self wt. plus other loads = w = 2 kN/m^2
Rise of the shell = h = 9.547 m
Span of dome = $2r_1$ = 58.78 m; 1/2 span = r_1 = 29.39 m
Upper ring radius = r_2 = 5.225 m

Reference	*Step*	*Calculations*
	1.	*Determine N_ϕ and N_θ* w = 2 kN/m; R = 50 m; ϕ_O = 6°; ϕ_K = 36°; P = 8 kN/m

1	2	3	4	5	6
ϕ	$\dfrac{wR(\cos\phi_c)}{\sin^2\phi}$	$\dfrac{P\sin\phi_O}{\sin^2\phi}$	N_ϕ (2+3)	$wR\cos\phi$	N_θ (5 − 4)
6°	0	76.5	76.5	99.4	22.9
30°	51.4	3.3	54.7	86.6	31.9
36°	53.7	2.4	56.1	80.9	24.8

(Reference column beside the rows 6° and 30°: Eq. (4.3), Eq. (4.4))

Reference	*Step*	*Calculations*
	2.	*Design of shell*
Eq. (4.3)		All forces are compression; Max value = 76.5 kN
		Max. comp. stress = $\dfrac{76500}{1000\times 60}$ = 1.3 N/mm^2
		This is very long. Allowable $\left(\dfrac{f_{ck}}{4}\right)$
		Check for buckling $\sigma_{cr} = \dfrac{Et}{R\sqrt{3}} = \dfrac{25\times 10^3\times 60}{50000\times\sqrt{3}}$ = 17.3 N/mm^2
		Assume FS = 4. Hence σ_{cr} (safe) = 4.3 N/mm^2
		Provide 0.3% steel
	3.	*Tension on bottom ring beam*
		$T = N_{\phi K}\cos\phi_K \times R$ = 56.1 × cos 36 × 29.39 = 1333 kN
Eq. (4.7)		[Design steel and section 1333 × 10^3/230 = 5795 mm^2. Use 20 nos. 20 mm = 6264 mm^2 and 500 × 900 ring beam]
		Check for stresses
	4.	*Compression on top ring beam*
Eq. (4.9)		$H = P\cot\phi_O \times r = \dfrac{8\times 5.225}{\tan 6°}$ = 397 kN
		Comp. allowed in concrete = 7 N/mm^2
		Area needed = $\dfrac{397\times 1000}{7}$ = 56700 mm^2
		Provide 200 × 300 mm ring beam

EXAMPLE 4.4 (Design of conical umbrella roof)

A conical umbrella roof is supported at the centre by a circular column and has the following dimensions [4]:

Height of conical roof = 1 m. Total column height = 1 + 3 = 4 m (Assume height below = 3 m)

Slope angle = 78°.41′ with vertical (< 30 with horizontal)

Shell thickness = 60 mm

Column diameter = 400 mm

Radius of shell from the centre of column = 5 m

Assume total load = 210 kg/m^2 = 0.21 t/m^2

(Let z = depth below the apex. Total z = 1 m)

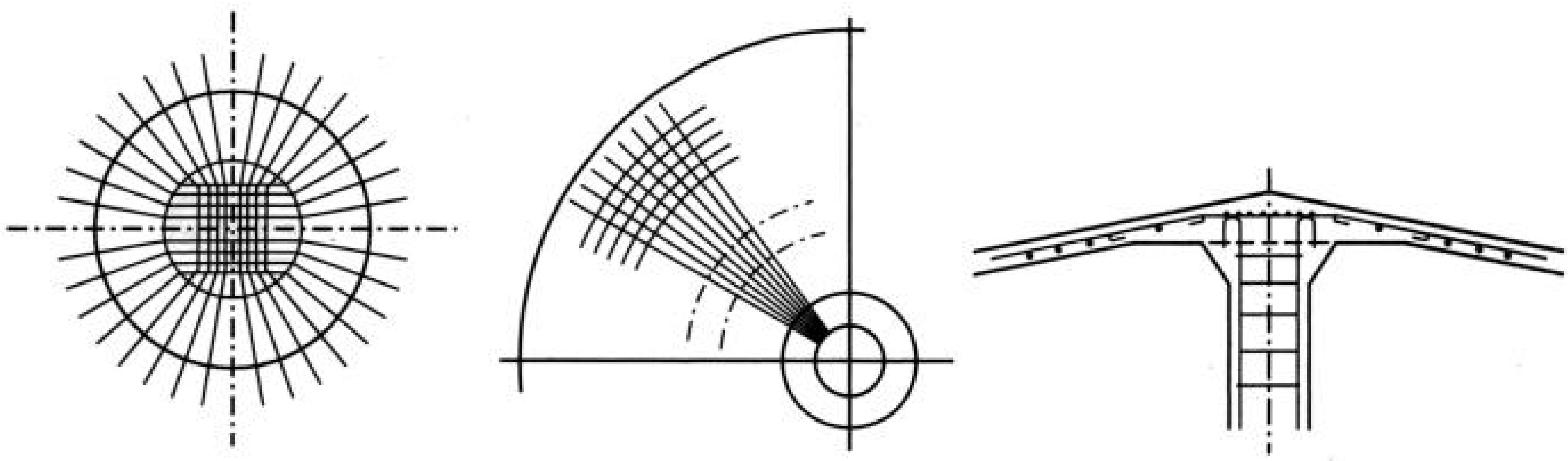


Figure E4.4 Detailing of conical umbrella roof.

Reference	*Step*	*Calculations*
	1.	*Divide depth into 6 dimensions 0 to 6 from top as shown* Results of calculation are as follows:

Point	z(m)	N_ϕ(t/m)	$N_\theta = 5.25z$
0	0	∞	0.0
1	0.1	–27.07	0.53
2	0.2	–13.12	1.05
3	0.4	–5.74	2.10
4	0.6	–2.91	3.15
5	0.8	–1.23	4.20
6	1.0	–0.0	5.25
		(Tension)	(Compression)

$\cos\alpha = 0.1961$; $\sec^2\alpha = 0.0385$; $\tan\alpha = 5.0$; $\tan^2\alpha = 25$

Reference	*Step*	*Calculations*
	2.	*Calculate N_ϕ (tension of meridinal forces)*
		$$N_\phi = \frac{w(h^2 - z^2)\sec^2\alpha}{2z}$$
		Point 6, $N_\phi = 0$
		Point 4, $N_\phi = \frac{0.21 \times (1 - 0.36)26}{2 \times (0.6)}$ = 2.91 t/m
	3.	*Calculate N_θ (compression, hoop forces)*
		$N_\theta = wz\ \tan^2\alpha = 0.21 \times 25 \times z = 5.25z$
		As the hoop forces are compressive, only minimum steel is required. Meridian forces are tensile. As it is infinite at zero radius, we flatter it at the column head. The structural layout is shown in Fig.
	4.	*Design of column*
		Height of column = 4 m
		Adopt diameter not less than 500 mm
		Design the column

EXAMPLE 4.5 (Design of conical dome supported on ring beam)
A conical dome roof is 10 m in diameter its height is 3 m and its semi vertical angle = 59°. Assume the slab thickness 100 mm. Analyze the shell [5].

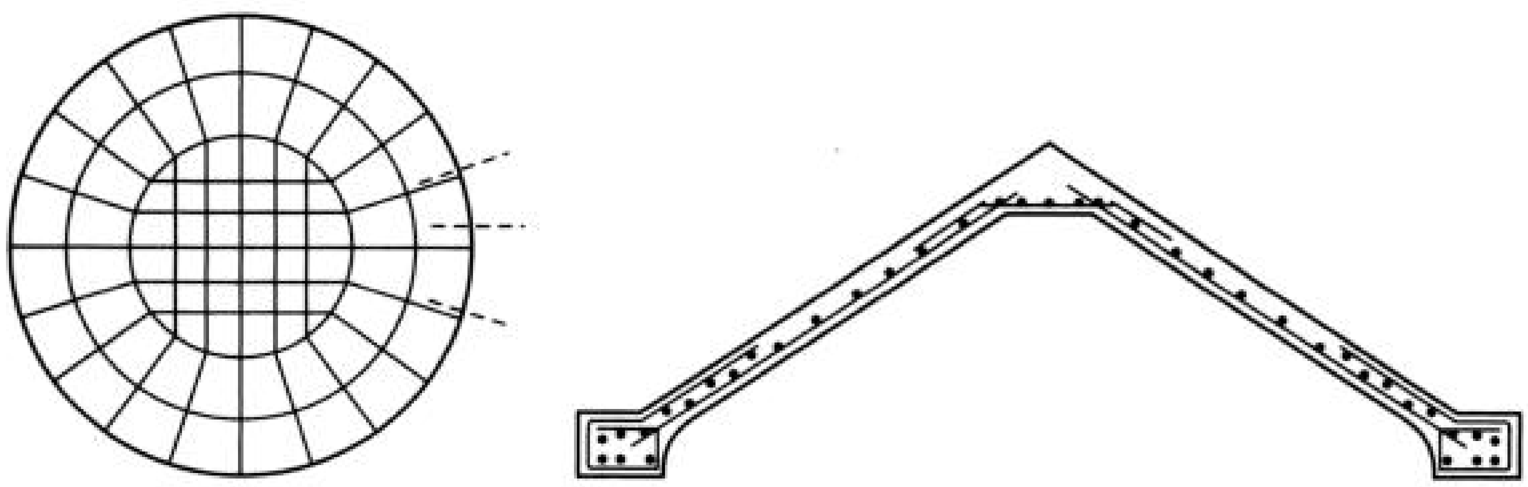


Figure E4.5 Detailing of conical dome supported as ring beam.

Reference	*Step*	*Calculations*
	1.	*Assume loading*
		Loading DL = $0.1 \times 1 \times 2400$ = 240 kg/m^2
		Insulation = 25 kg/m^2
		Live load = 40
		....
		305 say, 310 kg/m^2
		....

Reference	*Step*	*Calculations*
	2.	*Find expressions for meridian and hoop forces*
		1. Meridian force = $\dfrac{wz\sec^2\alpha}{2}$
		$N_\phi = \dfrac{310\times(\sec^2 59)}{2}\times z = (584.5)z \text{ kg/m}$
		2. Hoop forces = $wz\tan^2\alpha$
		$N_\theta = 310 \times (\tan^2 59) \times z = (858.6)z$ kg/m
	3.	*Tabulate the values at* z = 0, 1, 2, 3 m

z	M_ϕ (kg/m)	M_θ (kg/m)
0	0	0
1	585	859
2	1169	1717
3	1754	2576
	(Tension)	(Compression)

Reference	*Step*	*Calculations*
	4.	*Check compression*
		Max. stress = $\dfrac{2576}{1000\times1000}$ = 0.02 kg/mm^2 = 0.2 N/mm^2
		This is less than 2N/mm^2 allowed for M20 concrete.
		Provide normal 0.12% Fe 415 slab
	5.	*Tension in ring beam*
		$T = \dfrac{wh^2\tan^2\alpha\sec\alpha}{2}$. The value of T with h = 3 m
		$= \dfrac{310\times9\times(\tan 59)^2\sec 59}{2}$ = 181 kN = 7500 kg
	6.	*Design ring beam for this force*
		Design as given in Chapter 4 on spherical domes.
		Tension in concrete should also not exceed $f_{ck}/10$

REVIEW QUESTIONS

1. What are the advantages and disadvantages of a (a) semicircular dome and (b) segmental dome.
2. What is the difference in the assumptions of distribution of dead load and live load in domes and cylindrical shells? Indicate how wind loads are not of much importance in the design of the main surface of a dome.
3. Sketch the variation of meridinal stress and hoop stress in a dome due to (a) dead load and (b) live load.
4. Indicate the nature of forces in the following parts of a dome: (a) bottom ring beam and (b) top ring beam for a canopy for the top opening.
5. Indicate the types of shells made from a conical shell.
6. Explain the difference between the forces acting on (a) conical roof and (b) conical umbrella roof.
7. What is the need to provide a ring beam at the bottom of a conical umbrella roof?

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[3] Wilhelm Flugge, Stresses in Shells, Stanford University, 1983.
[4] Fisher, L., Theory and Practice of Shell Structures, Wilhelm, Ernst and Sohn, Berlin.
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5

ANALYSIS OF CIRCULAR CYLINDRICAL SHELLS

5.1 INTRODUCTION

Cylindrical shells are produced by a straight line generator moving over a given curve (directrix) at its two ends. Accordingly, we can have a circular cylindrical shell (with a circle as directrix), hyperbolic cylindrical shell (with a hyperbola as directrix), elliptic cylindrical shell (with an ellipse as directrix), and catenary cylindrical shell (with a catenary as directrix). In this chapter, we deal only with circular cylindrical shells.

Circular cylindrical shells are generally classified as long and short shells. The geometrical characteristics of these shells are shown in Figure 5.1. Long shells were very popular for roofs of factories and short shells for aircraft hangars, etc. As the formwork for these shells is rather expensive for factory roofs, long shells are now being replaced by folded slabs.

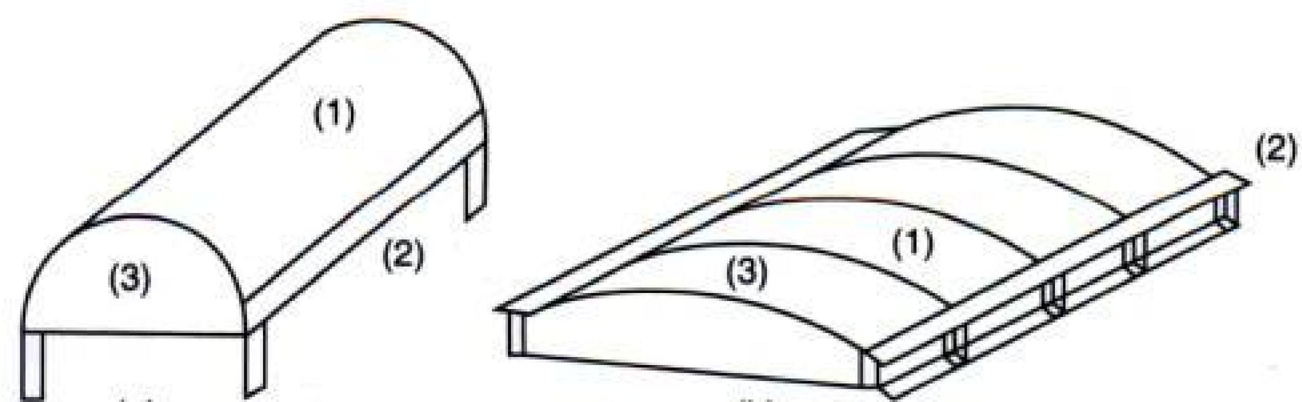

Figure 5.1 Classification of circular cylindrical shell: (a) Long shell with vertical edge beam, and (b) Short shell with horizontal edge beam; (1) Shell, (2) Edge beam, and (3) Transverse stiffners-beams or frames for long shells and arches for short shells.

Membrane analysis of circular cylindrical shells is very easy. For bending analysis of these shells, various investigators have presented different methods and equations. Closed form solutions are very useful for deriving computer based solutions. In fact,

there are many pieces of software available now for the analysis of these shells by computers. We have already dealt with the general theory of the differential equation used for the analysis of these shells in Section 3.4.

However, in this chapter, our main aim is to make a fundamental study of the problem, to understand intuitively the nature of the forces that act on these shells. The ASCE, Manual 31 on design of Concrete Shell Roofs method advocates a method similar to that we are familiar with and use for elementary analysis of indeterminate structures in theory of structures [1][2]. As it is easy to understand this method, we examine the method given in the ASCE Manual No. 31 in this chapter. It will give us a good idea of the nature of the forces in these shells so that when we use the modern computer software, we can have a better awareness of the output. It will also enable us to design ordinary reinforced concrete cylindrical shells not by using software but by using published tables. An analysis by hand computation is tedious and unreliable. For an approximate analysis of long cylindrical shells (where the length is greater than 3 times its radius), the beam method described in Chapter 6 can be used. Section 5.12 of this chapter gives the general recommendations for the layout of these shells. References [3] and [4] give additional tables (in addition to Manual No. 31) for the design of cylindrical shells. **(Regarding signs, we will follow the ASCE Manual No. 31 convention of tension as positive and compression as negative.)**

5.2 CLASSIFICATION OF CYLINDRICAL SHELLS

The following symbols describe the dimensions of the shells (Figure 5.2):

L = Span (the length of the shell)

B = Chord width

r = Radius of shell (ACI manual notation)

(In Chapter 4, for domes, radius = R and r = ½ span)

h = Rise above the edges or from top of edge beam

t = Thickness

ϕ_k = Half central angle

ϕ = Angle of any point measured from the right edge (valley) of shell

x = Distance from left support (diaphragm, also called traverse)

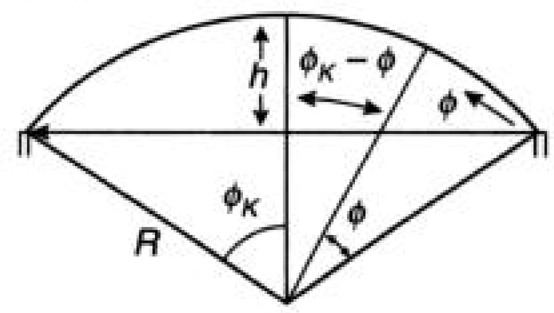

Figure 5.2 Circular cylindrical shell dimensions. (Note ϕ is measured from edge.)

Method 1: Considering the *structural action* of these shells as a beam, they are divided into three groups:

1. $L/r > 3$ (say, π)—Long barrel shells. These act as curved beams. They can be analyzed by the "beam method" described in Chapter 6.
2. $L/r < 0.5$—Short barrel shells. These act more like arches.
3. Intermediate types—$L/r > 0.5$ but < 3.

Method 2: *ASCE Manual No. 31* classifies cylindrical shells into the following two classes and separate tables have been derived for each. (We will use this criterion in our study.)

1. Shells with $r/L < 0.6$ or $L/r > 1.66$ (Type I shells). This means that the length of the shell is at least more than approximately 1.7 times the radius of the shell. In such cases, we can use ASCE Manual No. 31 for its analysis.
2. Shells with $r/L > 0.6$ or $L < 1.6r$ (Type II shells or short shells)

This classification is made on the observation that in type 2 shells, the length of the arc is so large that the effect of forces applied on one edge will not act on the other edge. On the other hand, in type 1 shells, the effect of forces applied on one edge will be felt on the other edge also. This classification is similar but simpler than the Jakobsen classification given below.

Method 3: The classification based on Aas Jakobsen's parameters for cylindrical shells, as stated in IS 2210–1988, is based on ρ and k, i.e.,

$$\rho = 8\sqrt{\frac{12\pi^4 r^6}{L^4 t^2}}, \quad k = \frac{\pi^2 r^2}{L^2 \rho^2}$$

If $\rho < 7$ and $k < 0.12$, the effect of forces from one side of the shell affects the other side. These are called *long shells* or Type I shells. If ρ exceeds 10 and k exceeds 0.15, the effect of forces from one side does not affect the other side. These are short shells or Type II shells.

Shells with ρ between 7 and 10, and k between 0.12 and 0.15 are relatively infrequent.

5.3 NOTATIONS USED FOR FORCES AND DISPLACEMENTS IN MANUAL NO. 31

[**Note:** According to ASCE Manual No. 31 sign convention, tension is taken as +ve and compression as –ve. Bending moment that produces tension in the inner fibres of the shell is taken as +ve.]

Angle ϕ is measured *from the springing of the shell* and ϕ_k is the half angle subtended by the shell. We will adopt these conventions. (In spherical shells, Chapter 4, ϕ was measured from the crown.)

1. Primary system (see Sec 3.2 also). The major forces acting on the cylindrical shell are as shown in Figures 5.2 and 5.3 and taken as follows:

[**Note:** Positive values of T_x and T_ϕ direct tension and negative values denote compression.]

T_x = Direct force (tension or compression) in the longitudinal direction

T_ϕ = Direct forces in the ϕ (transverse) direction

S = Tangential shear force considered +ve when it produces tension in the direction of increasing value of x and ϕ

M_ϕ = Bending moment in radial face is taken as +ve when it *produces tension inside the shell*

(We neglect all the other four stress resultants stated in Section 3.2.)

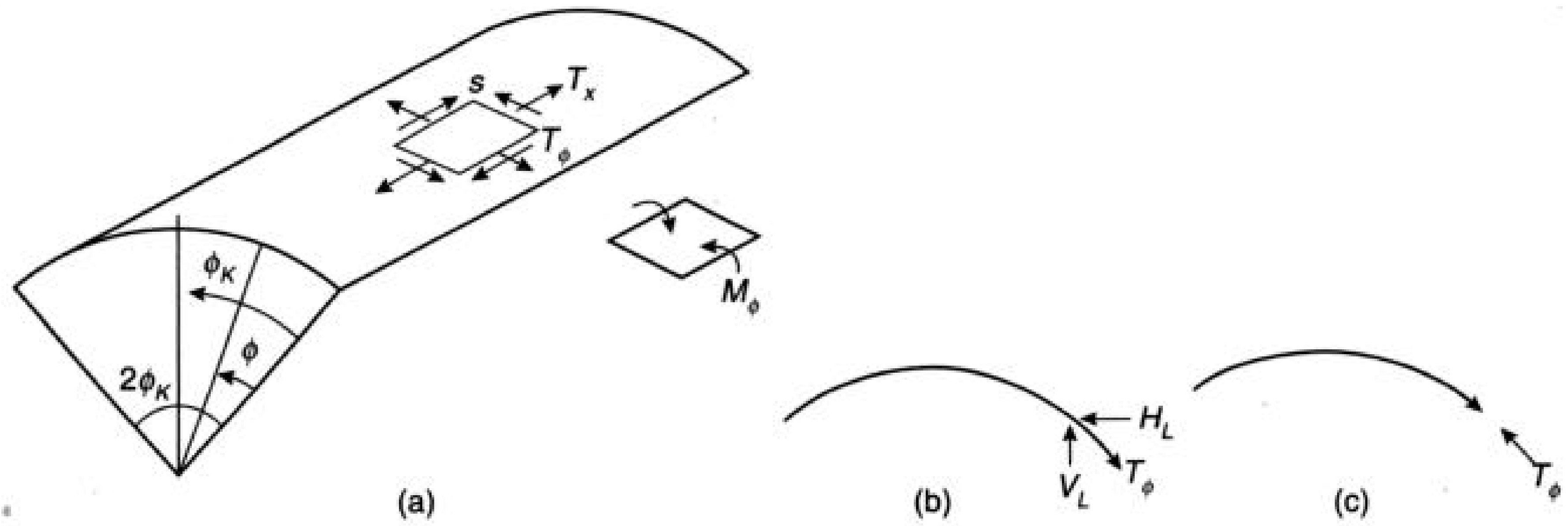

Figure 5.3 Forces considered in analysis and design of circular cylindrical shells: (a) T_x, $T_\phi \cdot S$ and M_ϕ, (b) Balancing of T_ϕ forces in long shells by H_L and V_L, and (c) Balancing T_ϕ in short shells.

2. Corrective line load. At the boundary of a cylindrical shell, we have to apply boundary forces to bring back equilibrium. The forces we apply are:

V_L = Radial line load applied along the longitudinal edge
H_L = Horizontal line load applied along the longitudinal edge
M_L = Transverse moment applied along the longitudinal edge
S_L = Shear load applied as line load
T_L = Tangential line load applied along the longitudinal edge
R_L = Radial line load

3. Displacements

ΔV = Vertical displacement
ΔH = Horizontal displacement
ΔL = Displacement in the longitudinal direction
u = Longitudinal displacement +ve in the increasing direction of x
v = Tangential displacement – +ve in the increasing direction of ϕ
w = Displacement in the radial direction – +ve in the outward direction

[**Note:** We can also reduce displacements v and w to ΔH and ΔV.]

5.4 ASCE MANUAL NO. 31 METHOD OF ANALYSIS

We briefly study the method without going into the derivations of the formulae, which are available in References [1] and [2].

The ASCE method is similar in principles to the classical method of analysis of indeterminate structures. We first make the structure determinate (membrane analysis) and then make corrections to satisfy the equilibrium of the original structure. In the case of cylindrical shells, we first find out the membrane forces (which are determinate) and then make corrections to satisfy the boundary conditions. In the correction analysis, we

have to use the bending theory of shells as bending moments are also produced when we apply correction forces on the boundary of the shell. (*Note:* Additional tables to those in Manual No. 31 are available in References [3] and [4]. Interpolation of tables is not valid.) As already stated in the Introduction, additional tables for designers have been published in References [3] and [4].

5.5 REPRESENTATION OF LOADS

Dead load on the shell acts uniformly along the circular surface. Live load is usually *assumed as UDL on the plan area.* IS 875 parts 2 to 4, 1987 (IS Code of Practice on design loads for buildings and structures) states that the minimum live load should be as follows. Taking h as the rise and B as the chord width, the live load is to be,

$$p_u = [75 - 345(h/B)^2] \text{ kg/m}^2$$

but not less than 40 kg/m^2.

Dead load is taken as uniform along the circumference of the shell and live load as uniform along the horizontal projection of the shell. As membrane analysis is determinate problem any load representation can be used for its analysis. *However, for the bending analysis of the shell, as we have to depend on differential equations, it is necessary to represent the load by its Fourier representation in the longitudinal direction.* This will make integrations and differentiation easy.

Thus, load is represented as shown in Fig. 5.4

$$p = \frac{4}{\pi} p' \sum_{n=1,3,5}^{n} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right)$$

Though taking together n = 1, 3 and 5 will give accurate results, for all practical purposes n = 1 gives good results. Thus, to make matters simple, we will take only the first term of the loading and represent load as shown in Figure 5.4. (*Note:* we enhance p' to $(4/\pi \times p')$ as given below.)

$$p = \left(\frac{4}{\pi} p'\right) \sin\left(\frac{\pi x}{L}\right)$$

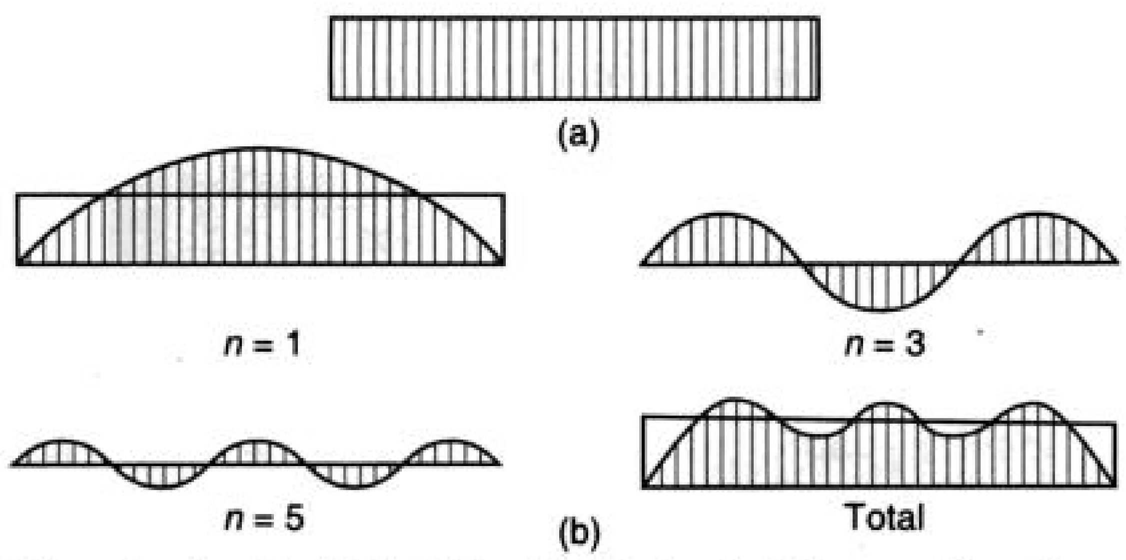

Figure 5.4 Representation of uniformly distributed load by Fourier load for eases in mathematical analysis: (a) UDL, and (b) Fourier loading with n = 1, 3 and 5 and resulting load.

5.5.1 Conversion of Live Load to Equivalent Dead Load

Reference [3] recommends conversion of live load into equivalent dead load for ease of calculation. Such conversion gives results which are on the safer side. We will follow this recommendation.

Unlike dead load which acts on the surface of the shell, the live load prescribed acts on the plan area of the shell as shown in Figure 5.5. In addition, dead load also produces greater stresses than live load. Comparisons made with different r/L values indicate that the effect of uniform live load can be converted into an equivalent dead load by the following formula giving conservative results [3]:

$$p_d = p_l\left(\frac{\sin \phi_k}{\phi_k}\right)$$

The value of LL is reduced by this expression.)

This will enable us to work with equivalent dead load and thus simplify our work. As live loads are usually small, we may also simply add them on the dead load.

In shell surfaces inclined at less than 40° to the horizontal, the wind load effects can also be suction. Hence, in many cases, we neglect the wind load in shell design.

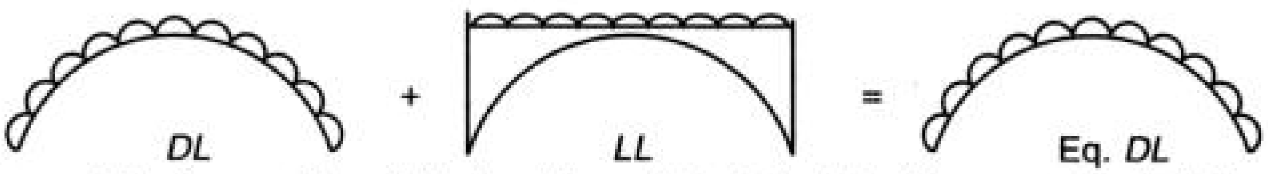

Figure 5.5 Conversation of live load to equivalent dead load for ease of calculations.

5.6 TABLES IN ASCE MANUAL NO. 31 FOR MEMBRANE FORCES AND DISPLACEMENTS

5.6.1 Use of ASCE Manual for Calculation of Membrane Forces

Membrane forces are only three in number: direct forces T_x and T_ϕ (which can be tension or compression) and (shear force). These can be easily derived from the three equations of statics. Similarly, the formulae for membrane displacements can be derived from the theory of elasticity. References [1] and [2] may be consulted for the derivation of the formulae for these forces.

However, these forces and displacements for uniform loading and sine loading can be easily calculated from Table 1 of ASCE Manual No. 31, reproduced in Appendix E of this book. Interpolation of values in Table 1 used for member are analysis is allowed.

The tables in the Manual for membrane analysis are:

Table 1A—Membrane forces and displacements for live and dead loads *uniformly distributed along the length of the shell.*

Table 1B—Membrane forces and displacements for live and dead loads with sine loading (n = 1) along the length of the shell.

Table 1C—Membrane forces and displacements for live and dead loads with sine loading (n = 3) along the length of the shell.

5.6.2 Distribution of Membrane Forces Across Cross-Section and Along the Length of a Small Circular Cylindrical Shell

Figure 5.6 gives the nature of the distribution of the maximum membrane forces for dead load and live load on *a circular cylindrical shell*. The maximum values for T_x and T_y occur at the centre ($x = L/2$) and shear at the ends ($x = 0$ and L).

1. With dead load, we can note the following:
   (a) With central angle $\phi = 90°$, both T_x and T_ϕ values will be nil and S value is maximum at the springing.
   (b) If we make the central angle less than 90° (we usually adopt $\phi = 30$ to 45 degrees only), T_x, T_ϕ and S will have definite values.
   (c) At the edges, T_x can be taken by the shell, but T_ϕ and S have to be balanced. If we provide an edge member, we can take care of the S forces, but T_ϕ has to be balanced. If we have no edge member, both T_ϕ and S have to be balanced. ASCE Manual No. 31, Tables 2 and 3, give us data for balancing these forces.
2. With live load, the values vary as shown in Figure 5.6.

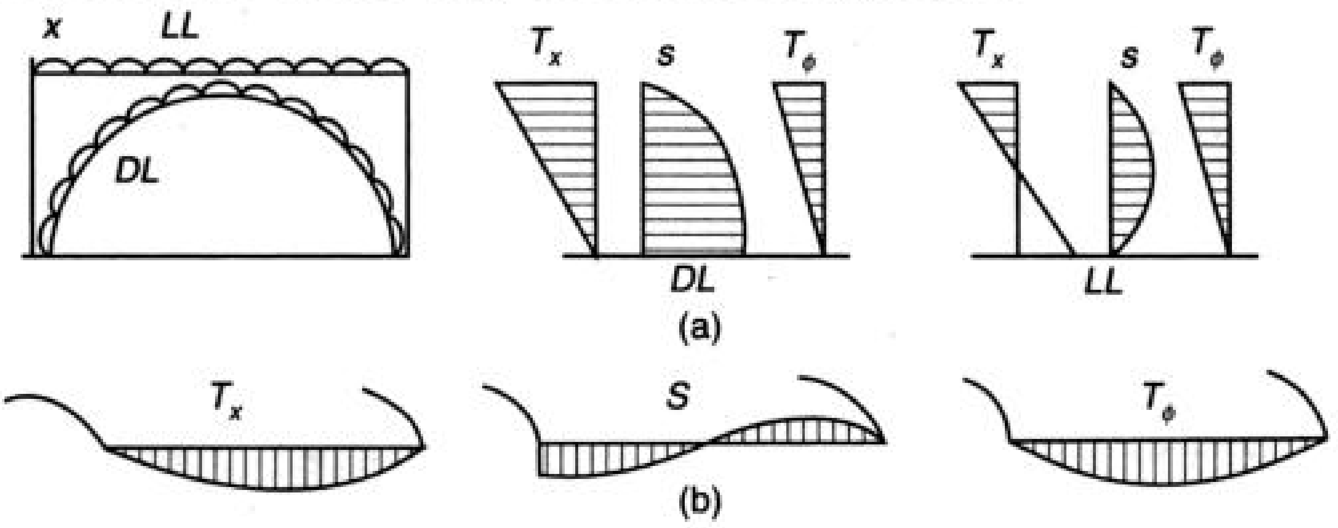

Figure 5.6 Distribution of membrane forces: (a) Distribution along cross section due to dead and live loads, and (b) Distribution along length for sine loading.

5.7 BENDING ANALYSIS OF CYLINDRICAL SHELLS (CORRECTION ANALYSIS)

The boundary conditions of a cylindrical shell require modification of the membrane forces due to the presence of boundary members which cannot be in equilibrium. For example, as shown in Figure 5.3, the membrane T_ϕ forces got by membrane analysis cannot exist at the boundary of a shell. Hence, we have to apply an equal and opposite T_ϕ force at the boundary to cancel its effect. We apply it as H_L and V_L for long shells and T_L for short shells. This will produce bending in the shell, and thus we have to depend on the bending theory for its solution. The results of bending theory for application of these forces are tabulated in Tables 2A for long shells and 3A for short shells in Manual No. 31.

We have already seen in Chapter 3 that if we take all the eight unknown forces, we end up with an eighth degree partial differential equation. Finsterwalder et al. [1] have made many simplifications to arrive at a solution to the problem. To simplify the problem,

ASCE published Manual No. 31 in 1952 giving simplified tables. Though, today, we have computer software to analyze these problems, in this book we will use Manual No. 31 to understand the fundamental behaviour of cylindrical shells. As already stated, the procedure adopted in this manual is the same as that we use in the structural analysis for the solution of indeterminate structures.

5.8 APPLICATION OF CORRECTIVE LINE LOADS

We have already seen that if ϕ_k is less than 90°, we have to balance the edge forces T_ϕ and S, as otherwise force equilibrium will not be maintained.

Now, to make the determinate structure (membrane analysis) satisfy boundary conditions in a cylindrical shell of definite size without edge members, we have to apply equal and opposite line loads to T_ϕ and S. The edges cannot sustain T_ϕ and S if there are no edge members.

We have seen that ASCE Manual 31 divides cylindrical shells into two types of shells—those with ($r/L < 0.6$) and with ($r/L > 0.6$). Different correction Tables (2A and 2B for long shells and 3A and 3B for short shells) have been devised for these shells. (Table A refers to forces and Table B refers to displacements.) This has been done for the following reason.

When r/L is 0.6 or less, the line loads applied on one edge create significant stresses and displacements at the opposite edge. Hence, Tables 2A and 2B have been compiled by simultaneous application of line load at both longitudinal edges. The points on the shell are denoted by ϕ, the angle from the edge of the shell. It varies from ϕ_k to 0. The line loads are applied as V_L, H_L, S_L and M_L.

When r/L is more than 0.6, the application of edge force on one side does not significantly affect the points on the opposite side. Hence, Tables 3A and 3B have been derived for one application of the line load on the edge. The points on the shell are represented by $s/\sqrt[4]{rtL^2}$, where s is the distance of point from the edge of the shell. It is tabulated from 3.2 to 0. In this case, the correction loads are applied as tangential load T_L, radial load R_L, shear S_L and moment M_L as shown in Figure 5.7.

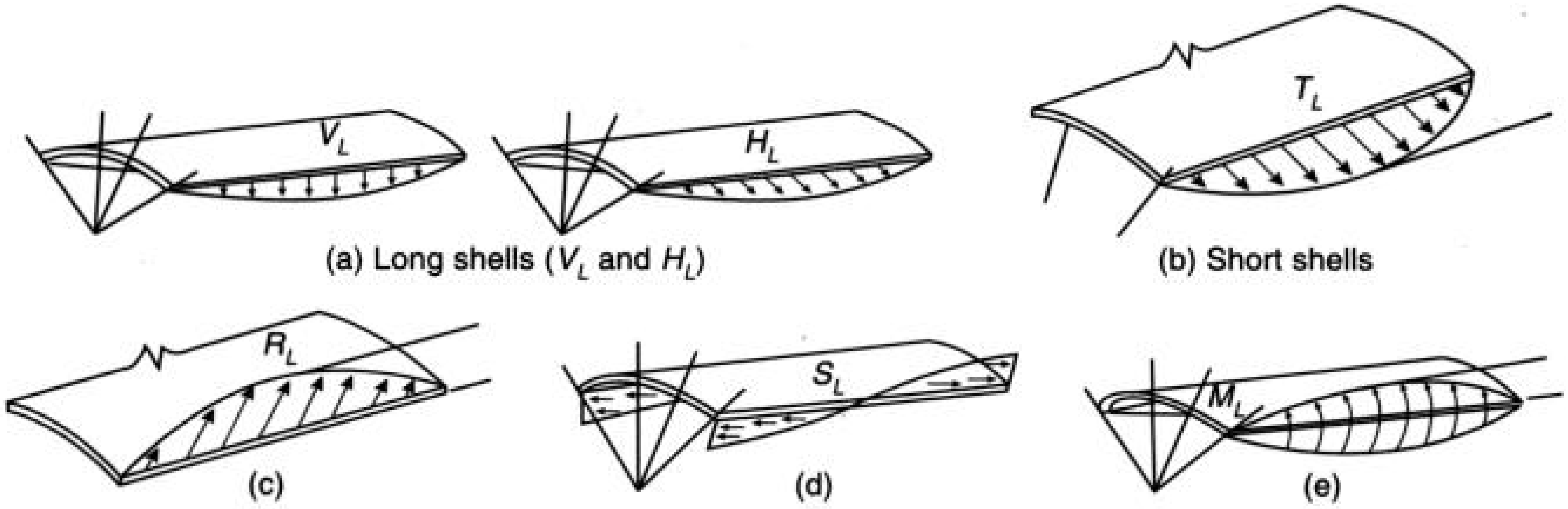

Figure 5.7 Application of correction loads along the length of the shell according to Tables in ASCE Manual No. 31. (a) Component of $T_\phi = T_L$ as V_L and H_L for long shells, (b) $T_\phi = T_L$ for short shells, (c) Line load R_L, (d) Line load S_L, and (e) line Load M_L.

5.9 DESCRIPTION OF CORRECTION TABLES

As already stated, Tables 2A and 2B, corresponding to long shells, and Tables 3A and 3B, corresponding to short shells, have been derived from the bending analysis of the shells. The correction application of T_ϕ will produce T_x, S and M_ϕ. The correction application of S will produce T_x, T_ϕ and M_ϕ. **Tables 2A and 3A give the forces produced and 2B and 3B, the displacements produced due to application of corrective loads.**

Table 2A is for correction analysis for T_x, T_ϕ, S and M_ϕ produced by various edge loads for,

$r/t = 100$ and $r/L = 0.1, 0.2, 0.3, 0.4, 0.5$ and 0.6 ($L = 10r$ to $1.66r$)

$r/t = 200$ and $r/L = 0.1$ to 0.6 (as above)

For various ϕ_k from 30 to 50.

Table 2B gives the coefficient for displacements ΔV, ΔH and θ for $r/t = 100$ and 200 and $r/L = 0.1$ to 0.6 due to application of V_L, H_L, S and M_ϕ.

Table 3A is to be used for correction analysis for T_L (corresponding to T_ϕ), S_L (corresponding to S) and radial load R_L for various rt/L^2. Similarly, Table 3B gives the displacements, v, w and θ, for various edge loads for short shells.

5.10 DESCRIPTION OF ANALYSIS OF SIMPLY SUPPORTED LONG SHELL WITHOUT EDGE BEAM ($r/L \le 0.6$)

Case (1) Long shells $r/L \le 0.6$ or $L > 1.67r$.

Step 1. Find membrane forces: By calculation or by use of Table 1, calculate the membrane forces T_x, T_ϕ, S for various points $\phi = \phi_k$ to $\phi = 0$ for $x = L/2$. It will be maximum at $\phi = \phi_k$ and decreases with value of $(\phi - \phi_k)$.

Step 2. Determine and apply corrective line loads: Membrane force T_x can exist at $\phi = 0$. But membrane forces T_ϕ and S present at $\phi = 0$ from membrane analysis if $\phi_k < 90°$ cannot exist. This has to be cancelled to satisfy edge condition, where $T_\phi = 0$ and $S = 0$. For long shells, apply line loads equal and opposite to T_ϕ and S as

$$V_L = T_\phi \sin \phi_k$$
$$V_H = T_\phi \cos \phi_k$$
$$S = S$$

The resulting correction values are obtained by Table 2A. Details of the method of correction are shown in Example 5.2.

Case (2) Short shells $r/L > 0.6$ or $L < 1.67r$.

Step 1. Find membrane forces as in Case (1).

Step 2. Determine and apply correction analysis.

Membrane forces T_ϕ and S at $\phi = 0$ have to be cancelled to satisfy the edge condition. Apply,

$$T_L = T_\phi \text{ (tangential edge load)}$$
$$S = S \text{ (shear edge load)}$$

The corrections are applied by means of Table 3A of Manual No. 31. Details of the correction are shown in Example 5.3.

[**Note:** The table for radial edge load R_L is used when we consider horizontal edge beams in short shells.]

Step 3 (for both Cases 1 and 2)—Find resultant forces
The sum of Steps 1, 2 and 3 gives the final forces, T_x, T_ϕ, and S per unit length. The value of M_ϕ is moment per unit length. The design of reinforcement and checking of stresses can be carried out in the usual ways (see Chapter 9).

5.11 ANALYSIS OF VARIOUS TYPES OF SHELLS

The different types of commonly used cylindrical shells are the following (as shown in Figure 5.8):

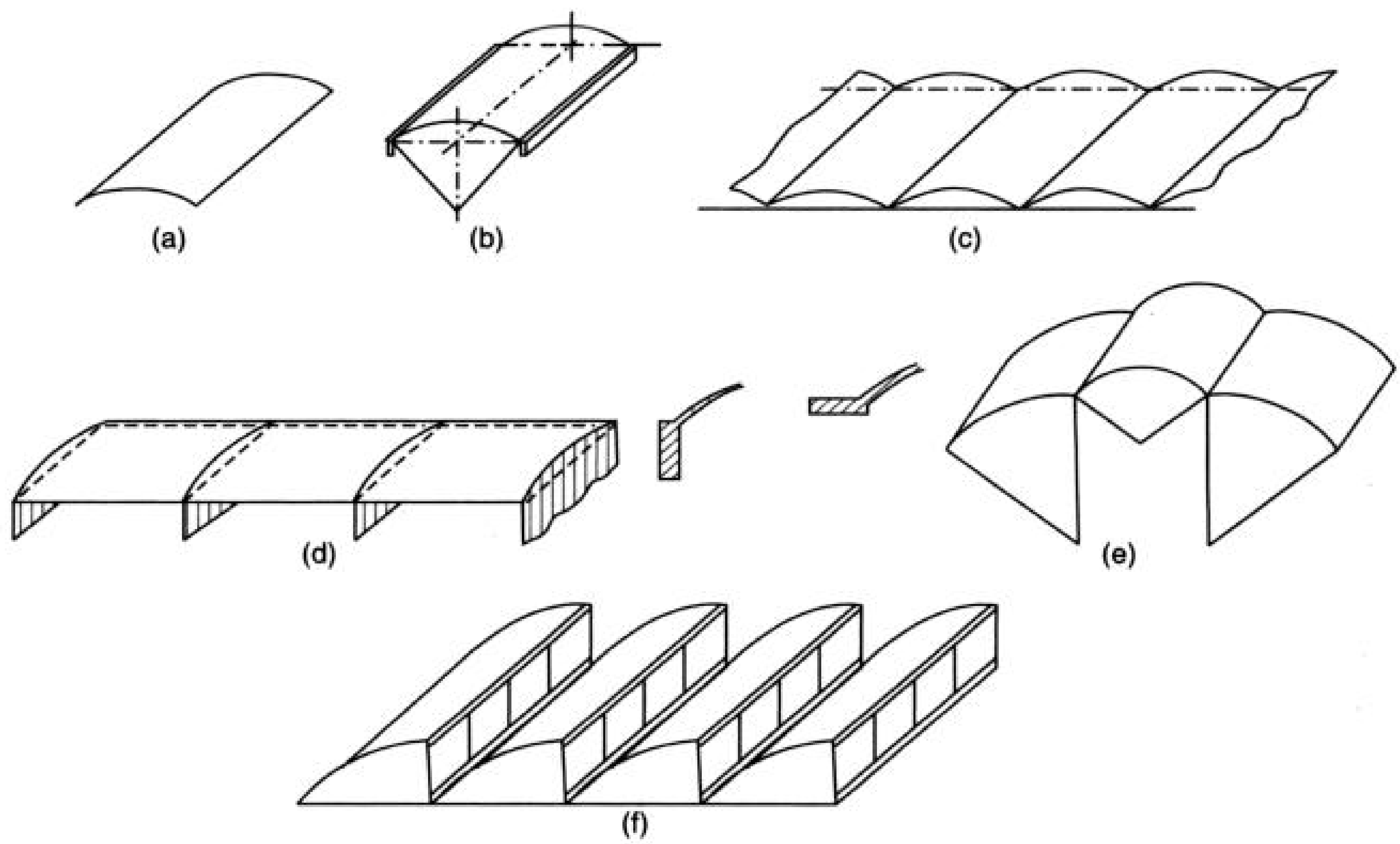

Figure 5.8 Types of cylindrical shells commonly used: (a) Single shell without edge beam, (b) Single shell with edge beam, (c) Multiple shell, (d) Continuous shells and vertical edge beam, (e) Combined shell formed by combination of shells and horizontal edge beam, and (f) Northlight shells.

1. Simple shells without edge beams (Figure 5.8a)
2. Shells with R.C. edge beams (Figure 5.8b)
3. Shells with prestressed edge beams
4. Multiple shells without edge beams (Figure 5.8c)
5. Multiple shells with edge beams
6. Continuous shells (Figure 5.8d)

We will deal with some of these types of shells separately in other chapters. It will be sufficient at this stage to know that, in these special cases, it will be necessary to satisfy displacement conditions in addition to equilibrium of forces. As an example, the special displacement conditions to be satisfied when we deal with shells with edge beams are explained in Chapter 8.

(One can omit Sections 5.12 and 5.13 on first reading and study the example worked out on the basis of the above descriptions.)

5.12 GENERAL PLANNING LAYOUT OF CYLINDRICAL SHELLS

Clause 7.2 of IS 2210—1988 "Criteria for Design of R.C. shell structures and folded plates" gives the following recommendations for planning of cylindrical shells:

1. The span should be less than 30 m unless prestressing is also planned.
2. The width of edge members should generally be limited to three times the thickness of the shell.
3. The radius of the shell should suit the acoustic requirements.
4. Single long cylindrical shells $L > 3B$ (length more than three times chord width) shall have a total depth between $L/6$ for small span and $L/12$ for larger spans. The rise of shells without edge beam should not be less than $L/10$.
5. For short shells (chord widths larger than three times the span), the rise of the shell shall not be less than the 1/8 the chord width. The chord width should not be more than six times the span as then it will act as an arch.
6. The semicentral angle should be between 30 and 40 degrees to facilitate easy concreting (without back forms).
7. The minimum cover shall be 13 mm or nominal size of reinforcement, whichever is greater.
8. The thickening at the junction of the beam should be of the order of 30% only, as cylindrical shells are singly curved shells and thickening is carried for a distance equal to 0.38 to $0.76\sqrt{rt}$, where r is the radius and t is the thickness of the shell.
9. The edges of the shell adjacent to the transverse are also thickened by increasing the thickness by 25 mm for a distance equal to 1/10th the length of the shell.

5.13 DIMENSIONING SHELL TO SUIT ANALYSIS BY ASCE MANUAL NO. 31

The following discussion is to show that the ASCE Manual No. 31 tables are restricted to be used for limited shell dimensions only. (Additional tables are available in References [3] and [4].) (This section can be omitted in the first reading).

Type I shells (Long barrel shells) Tables are available only for $r/t = 100$ and 200 and for $r/L = 0.1, 0.2, 0.3, 0.4, 0.5$ and 0.6.

Type II shells (Short shells) Tables are available only for $rt/L^2 = 0.002$ to 0.004, 0.006 to 0.010, 0.015 to 0.030 and 0.040 to 0.100.

(a) *Considerations for Type I shells.* We should first fix the half central angle ϕ_k between 30 and 40 so that the necessary curvature for shell action and for easiness of placement of concrete.

For example if we fix the angle as 35°, we get the relation as in Table 5.1 between radius r, chord width B, and rise of shell from springing h as follows.

TABLE 5.1 Relation between L, B, r and h for $f_k = 35°$

r/L	L/r (Available in Table)	B/r (Constant)	L/B	h/r (Constant)
0.1	10	1.147	8.72	0.181
0.2	5	1.147	4.36	0.181
0.3	3.33	1.147	2.90	0.181
0.4	2.50	1.147	2.18	0.181
0.5	2.00	1.147	1.74	0.181
0.6*	1.67	1.147	1.45*	0.181
Type II Shells	Type II Shells		Type II Shells	

*h = rise of shell from springing.

Another consideration is the thickness of the shell. It should not be less than 75 mm.

For 75 mm for $r/t = 100$; $r = 100 \times 0.075 = 7.5$ m

For 75 mm for $r/t = 200$; $r = 200 \times 0.075 = 15$ m

For 100 mm for $r/t = 100$; $r = 100 \times 0.10 = 10$ m

For 100 mm for $r/t = 200$; $r = 200 \times 0.10 = 20$ m

We have to choose specified parameters for using ASCE Manual No. 31 table. For example the chord width for $\phi_k = 35°$ is ($1.147 \times$ the radius). It is also advisable for Type I shells with edge beam to make the value of the total depth of the shell measured from the crown of the shell to the bottom of the edge beam as $L/10$. However, as the width of the shell is increased, the depth of the edge beam can be reduced. In the limit for a short shell, the edge beam can be made horizontal.

Example for choosing shell dimensions of long shell to suit Manual No. 31

As an example, let us take a 75 mm thick shell with $r/t = 100$ and $\phi_k = 35°$. Then $r = 100 \times 0.075 = 7.5$ m.

Let us choose table for $r/L = 0.3$. Then if you choose $r/L = 0.3$, $L = 7.5/0.3 = 25$ m

The shell will have $L = 25$ m; $r = 7.5$ m, $t = 75$ mm and $\phi_k = 35°$

The chord width $B = 1.147 \times 7.5 = 8.6$ m

Value of $h = 0.181r = 1.35$ m. We choose a shell with total depth 1/10 span (see Section 5.13).

Total depth = 25/10 = 2.5 m

Depth of edge beam = 2.5 – 1.35 = 1.15 m

Width of edge beam $3t = 3 \times 75 = 225$ mm

Preliminary dimensions can be as follows:

Span 25 m; radius 7.5 m; thickness 75 mm; edge beam 1150 × 225 mm

(b) *Considerations for Type II shells (Short shells).* The parameter for the analysis of short shells is rt/L^2 and tables are given for fixed values 0.002, 0.003, 0.004, 0.006, 0.008, 0.0010 to 0.10. If $\phi_k = 35°$, these are meant for the analysis of shells with r/L greater than 0.6 or L/B less than 1.45.

Let $t = 100$ mm, then $rt = 0.1r$

For $$\frac{rt}{L^2} = \frac{0.1r}{L^2} = 0.002 \text{ the least value}$$

we have $$L = \sqrt{\frac{0.1r}{0.002}} = 7.07\sqrt{r}$$

For $$\frac{rt}{L^2} = \frac{0.1r}{L^2} = 0.010 \text{ the highest value}$$

we have $L = \sqrt{r}$

Thus, the higher values of rt/L^2 are for smaller spans and shells of larger radius.

[**Note:** References [3] and [4] give additional tables for design of cylindrical shells by the Manual No. 31 method.]

SUMMARY

Cylindrical shells are classified into long shells or barrel shells and short shells. ASCE Manual No. 31 classifies them into Type I and Type II shells. This chapter explains the use of ASCE Manual No. 31 for the analysis of these shells. Reference [5] also gives a good account of the use of ASCE Manual. More theoretical methods are discussed in References [6] and [7].

It should be clearly understood that even though interpolation of values of Table 1 of Manual No. 31 is allowed we are not allowed to interpolate the values given in the ASCE Manual tables 2 and 3 for shells that we may arbitrarily plan. We should always plan shells in such a way that their dimensions suit the table in Manual No. 31. References [3] and [4] give additional tables which supplement ASCE Manual No. 31.

EXAMPLE 5.1 (Design of a Semicircular Shell Roof Resting on a Wall)
Design a semicircular reinforced concrete shell roof (vault) resting on the walls of a room 12 m in breadth and 20 m in length. Assume it is 100 mm thick and is carrying a dead load of 2.5 kN/m^2 and live load of 0.5 kN/m^2.

Procedure: From inspection of Table IA of Appendix E and Figure 5.6, we can see that for a vault with $\phi = 90°$, the value of T_ϕ and T_x for dead load is zero. But T_x has a value for the small live load only. Hence, we need no correction at the ends if we provide a small edge beam to take care of the shear. Hence, we need not use sine loading. The edge beam will be supported by the wall so that the beam will not be under bending.

(The shell is a semicircular vault T_x at edge due to DL = 0. Hence, no correction of T_x the major forces at the edge is required, we need not use correction tables Table 2A of Manual No. 31.)

Reference	*Step*	*Calculations*
		Shell data – Radius $r = 6$ m; Length $L = 20$ m; $L/r > 1.6$, Long shell
		$DL = 2.5$ kN/m^2 and $LL = 0.5$ kN/m^2
		Membrane analysis with non-Fourier loading
	1.	*Calculate max. value of T_x at crown* $(\phi_k - \phi) = 0$ at $x = L/2$
Appendix E		$T_x = \dfrac{pL^2}{r}\left(\dfrac{x}{L}\right)\left(1-\dfrac{x}{L}\right) \times \text{coeff.}$
Table IA		T_x due to dead load $(DL) = \dfrac{1}{4}(66.6) \times 2.5 \times 1 = 41.6$ kN (comp.)
		T_x due to live load $(LL) = \dfrac{1}{4}(66.6) \times 0.5 \times 1.5 = 12.5$ kN (comp.)
		Total compression = 54.1 kN/m (thickness 100 mm)
		Max. stress $= \dfrac{54.1 \times 1000}{1000 \times 100} = 0.54$ N/mm^2
		(Thickness of shell = 100 mm)
	2.	*Calculate T_x at* $(\phi_k - \phi) = 90°$ at $x = L/2$
		Due to $DL = 0$
		Due to $LL = \dfrac{1}{4}(66.6 \times 0.5 \times 1.5) = 12.5$ kN (tension)
		Stress $= \dfrac{12.5 \times 10^3}{1000 \times 100} = 0.12$ N/mm^2
		Max. tension = 0.12 N/mm^2 (which is small)
	3.	*Calculate S at crown for $x = 0$ (max. shear)*
		Due to dead load + live load = $pL \times$ (Coeff.)

Reference	*Step*	*Calculations*
Appendix E Table IA	4.	*Calculate maximum value of S at* $(\phi_k - \phi) = 90°$ *at edge (Figure 5.6)* Due to $DL = 2.5 \times 20 \times 1.0 = 50$ kN (max. shear) Due to $LL = 0$. Max. stress = 0.50 N/mm^2 [**Note:** Maximum shear due to LL occurs at $(\phi_k - \phi) = 45°$, but its value is less than the above value.]
	5.	*Calculate maximum value of* T_ϕ *at crown at* $x = L/2 = pr \times$ (coeff.)
Appendix E Table 1A		[**Note:** This does not depend on x value and is same over the length.] Due to $DL = 2.5 \times 6 \times 1 = 15$ kN/m (comp.) Due to $LL = 0.5 \times 6 \times 1 = 3$ kN/m (comp.) Total $T_\phi = 18$ kN/m; Stress = 0.18 kN/mm^2 (comp.)
	6.	*Calculate minimum* T_ϕ *value at* $(\phi_k - \phi) = 90°$ Due to DL; Due to $LL = 0$
	7.	*Summary of results* (stress in N/mm^2)

	At crown	*At edge*
Stress T_x	0.51 (comp.)	0.12 (tension) N/mm^2
Stress T_ϕ	0.18 (comp.)	0
Stress S	0	0.50

Reference	*Step*	*Calculations*
		As T_ϕ is zero at edge (valley), we need not undertake correction analysis. At the edge we will provide an edge beam to take care of the shear and small tension T_x produced by live load only.
	8.	*Design of shell* All the stresses are low. Provide nominal steel of 0.2 to 0.3% steel both ways in the shell.
	9.	*Design of edge for shear varying from 0 at* $x = L/2$ *to* 50 kN *at* $x = 0$ *and L (ends of the shell). Assume tension produced is equal to the shear.* Total tension = $\frac{L}{2} \times$ Average shear = $\frac{20}{2} \times \frac{50}{2}$ = 250 kN $A_s = \frac{250 \times 1000}{240} = 1042 \text{ mm}^2$ Provide a beam with the above steel at the ends to take care of this tension. If the beam is supported on walls, we need not provide for bending as in the case of shells with unsupported edge beams. (Provide 500 deep 200 mm wide beams with 6 nos. 16 mm bars giving 1206 mm^2. Check for stresses in concrete for tension.)

EXAMPLE 5.2 [Analysis of ASCE Type I (Long) shell without edge beam] ($\phi_k < 90°$)

A simply supported cylindrical shell has the following dimensions:

Radius of circle(r) = 7.5 m; Length of shell (L) = 15.0 m

$$\frac{r}{L} = 0.50 < 0.6 \text{ (Type 1 shell; } \frac{L}{r} = 2 > 1.66)$$

Half central angle (ϕ_k) = 40° (0.6977 radians)

$$\text{Thickness } (t) = 75 \text{ mm; } \frac{r}{t} = \frac{7500}{7.5} = 100$$

Procedure: As ϕ_k is less than 90°, we will have unbalanced edge forces. T_x forces can exist at the edge but not T_ϕ and S. It is necessary to make correction analysis for T_ϕ and S. As it is found that the live load can be converted to an equivalent dead load, we will consider the total load only as dead load. The loading will be Fourier load with the first term only. The analysis is carried out for maximum values of T_x and T_ϕ, which occur at $x = L/2$, and maximum value of S, which occurs at $x = 0$ or L.

Part I—Membrane Analysis: We first make four separate tables each for T_x, S, T_ϕ and M_ϕ. Carry out a membrane analysis. The operations are carried out in a tabular form. Tables 1 to 4 given at the end of this example are used to record the results of the calculations of T_x, S, T_ϕ and M, respectively.

Part II—Correction analysis: As forces T_ϕ and S obtained by membrane analysis cannot act at the edges, we have to make correction analysis using Table 2A. We apply correction line loads at the edges. That for T_ϕ will be,

$$V_L = T_\phi \sin \phi_k \text{ and } H_L = T_\phi \cos \phi_k$$

Data for analysis is entered in the Tables 1 to 4 pages 65, 66 as we proceed with the calculations.

For example, Table 1 of this example is tabulation of values for T_x forces which is maximum at $x = L/2$. The first line represents the T_x membrane force calculated by use of Manual No. 31 for various angles ϕ (corresponding to Table 1B of Manual No. 31).

The second line of Table 1 is correction due to V_L (**vertical correction component of T_ϕ**). Here the values will be a constant × coefficient taken from Table 2A of Manual No. 31.

The third line is the correction due to H_L (**horizontal correction component of** T_ϕ), the value of which is a constant × coefficient from Table 2A of Manual No. 31.

The fourth line is the correction for shear S which has to be compensated. Here also, correction is a constant × coefficient from Table 2A of Manual No. 31. The sum of the membrane analysis and the corrections give us the final values for design.

Similarly, we will have Table 2 of this example for shear(S) values and Table 3 for T_ϕ values.

There is no moment from membrane analysis. But as the correction analysis induces moments, we will also have a separate table for M_ϕ. Table 4 of this example is for transverse moment M_ϕ. In this table, there will be no membrane forces but we tabulate the M values produced by T_x, T_ϕ and S.

[**Notes:**

1. In this example as we have to do correction analysis, we will adopt sine loading Table 1B of Manual No. 31 (Appendix E) for membrane analysis and use Table 2A of Manual No. 31 for correction analysis.
2. We will use kg/cm^2 instead of N/mm^2 for convenience in manual calculation (we will be dealing with larger numbers with kg/cm^2).
3. We represent tension as +ve and compression as –ve as in Manual No. 31.]

Reference	*Step*	*Calculations*
	1.	*Calculate rise, chord length, etc.*
		B = Chord length = $2 \times r \sin 40 = 9.65$ m
		h = Rise of shell = $r(1 - \cos 40) = 1.755$ m
		Rise/breadth ratio = $\dfrac{1.755}{9.65} = \dfrac{1}{5.5}$
		Rise span ratio = $\dfrac{1.755}{15.5} \approx \dfrac{1}{8.6} > \dfrac{1}{10}$ (see Section 5.12)
	2.	*Calculate loads on shells*
		Dead load = $0.075 \times 2000 = 150$ kg/m^2
Sec. 2.7		Add LL plus weather proofing = 15 kg/m^2
		Total = 165 kg/m^2
Sec. 5.5		Live load = $75 - 345\left(\dfrac{R}{B}\right)^2$ kg/m^2 and not less than 40 kg/m^2
		$= 75 - 345\left[\left(\dfrac{1.755}{9.15}\right)^2\right] = 62.3$ (say, 65 kg/m^2)
Sec. 5.5.1		Use equivalent DL = $65 \times \left(\dfrac{\sin 40}{\phi_k}\right) = \dfrac{65 \times \sin 40}{0.6377} = 65$ kg/m^2 (approx.)
		Total load = 165 + 65 = 230 kg/m^2
		For sine loading increment factor = $4/\pi$ [We use the first term of Fourier loading.]
		Load = $\left(\dfrac{4}{\pi}\right) \times 230 = 293$ kg/m^3 (say, 295 kg/m^2)
		Part I Membrane Analysis
	3(a).	*Tabulate membrane force T_x at $x = L/2$ [see Table Ex. 5.2(1)]*
		$$T_x \text{ at } x = \frac{L}{2} = \underbrace{pr\left(\frac{L}{r}\right)^2}_{\text{constant}} \underbrace{\left[\cos(\phi_k - \phi) \times \frac{2}{\pi^2}\right]}_{\text{coeff. for } \phi}$$

Reference	*Step*	*Calculations*
		Coefficient refer to Table IB of Manual No. 31
		Constant = 295 × 7.5 × 4 = 8850 and
		$\cos(\phi_k - \phi)\left(\frac{2}{\pi^2}\right)$ for $(\phi_k - \phi) = 0$ equals 0.2026 as in Table IB, Appendix E.
Coeff. from		Hence, T_x at $(\phi_k - \phi) = 8850 \times$ (Coeff. for $(\phi_k - \phi)$
Appendix E		The coefficient can be calculated manually from formulae or taken from Table (Appendix E).
Table 1B		[**Note:** The values are tabulated in Table 1 for T_x given below.]
Coeff. from	3(b).	*Tabulate membrane force S at x = 0 [see Table Ex. 5.2(2)]*
Appendix E		$S = pr\frac{L}{r}\left[\sin(\phi_k - \phi) \times \frac{2}{\pi}\right]$ can be read off from Table in Manual No. 31 or calculated.
Table IB		S at edge $\phi = 0$ or $(\phi_k - \phi) = 40°$ is 4425 × 0.4092 = 1810
		[Coeff. in Table 2]
Coeff. from	3(c).	*Tabulate membrane force T_ϕ at $x = L/2$ [see Table Ex. 5.2(3)]*
Appendix E		$$T_\phi = pr[\cos(\phi_k - \phi)]$$
Table IB		Constant = pr = 295 × 7.5 = 2213
		Coeff. = $\cos(\phi_k - \phi) = 1$ for $(\phi - \phi_k) = 0$
		This can be read off from Table of Manual No. 31 or calculated manually.
	3(d).	*Value of M_ϕ*
		There is no bending in membrane analysis, but this will be produced when we apply correction edge forces.
		These calculations are made for $(\phi - \phi_k)$ values 0, 10, 30 and 40. (See Tables at the end of this example.)
		Part II Correction Analysis using Table 2A of Manual No. 31
	4.	[**Note:** Membrane analysis results in edge forces T_ϕ and S. Now we apply edge correction to correct the T_ϕ and S forces. They should be zero as there are no edge members to take these forces. T_x does not need any correction as the shell edge can take T_x forces.]
	4(a).	*Calculate edge correction loads*
		Value of T_ϕ at $\phi = 0$ or $(\phi_k - \phi) = 40$; $T_\phi = 2213 \times \cos 40 = 1695$
		This T_ϕ cannot exist as the edge is discontinuous. Hence, we apply equal and opposite correction forces as V_L and H_L as for long shells.
		$$V_L = T_\phi \sin \phi_k = 1695 \times \sin 40 \approx 1090$$
		$$H_L = T_\phi \cos \phi_k = 1695 \times \cos 40 \approx 1290$$

Reference	*Step*	*Calculations*
	4(b).	*We tabulate T_x at $x = L/2$ in Table Ex. 5.2(1); S in 6.2 for S at $x = 0$ in Table Ex. 5.2(2), T_ϕ at $x = L/2$ in Table Ex. 5.2(3) and M_ϕ at $x = L/2$ produced by V_L, H_L and S.*
		[**Note:** (1) Moment producing tension on the inside the shell is taken as +ve and tension on the surface as –ve.
		(2) As there is no edge beam, T_ϕ and S should be zero at the edges of the shell.]
Table 2A		(1) For T_x caused by application of V_L, H_L and S.
Appendix E		T_x due to corrective load $V_L = V_L \times (L/r)^2 \times$ Coeff. = 1090 × 4 × Coeff.
		T_x due to corrective load H_L = 1290 × 4 × Coeff.
		T_x due to corrective load S = 1810 × 4 × Coeff.
		(2) For S caused by application of V_L, H_L and S.
		S due to corrective load V_L = 1090 × 2 × Coeff.
		S due to corrective load H_L = 1290 × 2 × Coeff.
		S due to corrective load S = 1810 × 2 × Coeff.
		(3) For M_ϕ (There is no moment in membrane analysis.) due to V_L, H_L and S.
		M_ϕ due to corrective load $V_L = V_L \times r \times$ Coeff. = 1090 × 7.5 × Coeff.
		M_ϕ due to corrective load H_L = 1290 × 7.5 × Coeff.
		M_ϕ due to corrective load S = 1810 × 7.5 × Coeff.
	5.	*Tabulation is as follows:*

Table 1 (Example 5.2) Distribution of T_x at $x = L/2$ (in kg/m) for various ϕ values (Using Table 2A of Manual No. 31; $r/t = 100$; $r/L = 0.5$; $\phi_k = 40°$)

No.	T_x	$\phi_k - \phi$	0	10	20	30	40
		ϕ	40	30	20	10	0 (Edge)
1	Membrane load 8850 × Coeff. from Table IB		–1793	–1766	–1685	–1552	–1373
	Line loads						
2	V_L = 1090, 1090 × 4 × Coeff.		+23858	+6121	–29042	–25400	+113880
3	H_L = 1290, 1290 × 4 × Coeff.		–22110	–9497	+17227	+21668	–62846
4	S = 1810 × 4 × Coeff.		–848	–1019	–617	+3054	+13813
	Total		–893	–6161	–14117	–2230	+63579

[**Note:** Value of T_x at edge is not zero.]

Example. Calculation for V_L. Table 2A of Manual No. 31, for applying V_L we get coeff. for T_x @ $\phi = 40° = +\ 5.472$.

Hence, T_x = 1090 × 4 × 5.472 = +23858 (as in Table above)

Reference	*Step*	*Calculations*

(*Comments*. As a digression let us apply the beam theory to this shell using Table 6.2 page 78. We get;

$T_x = \frac{L^2}{r} \times [p \times \text{Coeff.}]$; Coeff. for $\phi_k = 40°$ for $\phi = 0$ at valley is 3.928

$$T_x = \frac{(15)^2}{7.5} \times 295 \times 3.928 = 34672 \text{ kg/m}$$

This value is much different from the theoretical value showing that beam theory is strictly applicable to long shells $L/r > 3$ only. However, if we consider total tension below the neutral axis for which we will provide steel, the result will not be significantly different.

Table 2 (Example 5.2) Distribution of Shear S **at** $x = 0$ **(in** kg/m**) for various** ϕ **values** (Using Table 2A of Manual No. 31)

No.	S	$\phi_k - \phi$	0	10	20	30	40
		ϕ	40	30	20	10	0
1	Membrane 4425 × Coeff.		0	–489	–694	–1408	–1810
2	$V_L = 1090$, 1090 × 2 × Coeff.		0	+4866	+1650	–7637	0
3	$H_L = 1290$, 1290 × 4 × Coeff.		0	–4877	–3842	+2626	0
4	$S_L = 1810$, 1810 × 4 × Coeff.		0	–303	–618	–333	+1810
	Total		0	–1303	–3774	–6752	0

[**Note:** Value of shear S at $\phi = 0$ (edge) is zero.]

Table 3 (Example 5.2) Distribution of T_ϕ **at** $x = L/2$ **(in** kg/m**) for various** ϕ **values** (Using Table 2A of Manual No. 31)

No.	T_ϕ	$\phi_k - \phi$	0	10	20	30	40
		ϕ	40	30	20	10	0
1	Membrane 2213 × Coeff.		–2213	–2179	–2079	–1916	–1690
2	$V_L = 1090$, 1090 × Coeff.		–309	–1165	–2248	–1323	+700
3	$H_L = 1290$, 1290 × Coeff.		–284	+469	+1828	+1973	+990
4	$S_L = 1810$, 1810 × Coeff.		–140	–106	+5	+141	0
	Total		–2946	–2981	–2494	–1125	0

[**Note:** Value of T_ϕ at $\phi = 0$ (edge) is zero.]

Reference	*Step*	*Calculations*

Table 4 (Example 5.2) Distribution of M_ϕ at $x = L/2$ (in m.kg/m) for various ϕ values (Using Table 2A of Manual No. 31)

No.	M_ϕ	$\phi_k - \phi$	0	10	20	30	40
		ϕ	40	30	20	10	0
1	Membrane		0	0	0	0	0
2	$V_L = 1090$, 1090 × 7.5 × Coeff.		–1082	–1107	–1025	–614	0
3	$H_L = 1291$ 1290 × 7.5 × Coeff.		+660	+730	+793	+574	0
4	$S = 1810$, M_f = 1810 × 7.5 × Coeff.		–46	–46	–19	+4	0
	Total		–468	–423	–251	+36	0

1. Variation of T_x along length is as sin $\pi x/L$
2. Variation of S along length is as cos $\pi x/L$
3. Variation of T_ϕ along length is as sin $\pi x/L$
4. Variation of M_ϕ along length is as sin $\pi x/L$

EXAMPLE 5.3 [Analysis of type 2 (short) shell without edge beam]

Analyze the forces in the shell of the following dimensions for the live load specified by IS:

Radius	$r = 22$ m	$r/L = 2.09 > 0.6$ (**type 2 shell; use Table 3A)**
Length	$L = 10.5$ m	$rt/L^2 = 0.015$; $L/r = 0.47$ (< 1.66)
Thickness	$t = 75$ mm	$(rtL^2)^{1/4} = 3.673$
Chort width	$B = 27$ m	$L/r = 10.5/22 = 0.477$

$$\sin \phi_k = \frac{27}{2} \times \frac{1}{22}. \text{ Hence, } \phi_k = 37.85°.$$

Assume equivalent total dead load = 230 kg/m^2.

Assume sinusoidal load as $\frac{4}{\pi} \times 230 = 295$ kg/m^2 as in Example 5.2.

[**Note:** In the case of short shells, the first two terms of the load $= \frac{4}{\pi} p_w \sum_{1,2,5}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L}$, i.e. $n = 1$ and 3 should be taken into account for *an accurate analysis*. In the present case, we will take only the first term as the object is only to illustrate the method. Table 3A gives correction values for $n = 1$ and $n = 3$ in separate tables. We will use table for $n = 1$ only.]

The characteristic of the shell is defined by the expression $\frac{rt}{L^2} = \frac{22 \times 0.075}{(10.5)^2} = 0.015$ (Use Table 3A and $rt/L^2 = 0.015$) given on page 322.

Procedure

Step 1: As in Example 5.2, carry out membrane analysis to find the forces in the determinate structure. Use Table IB of Manual No. 31 given in Appendix E. (Values of T_x, T_ϕ and S can also be found using the equations for membrane analysis. Interpolation of values in Table I of Manual No. 31 is allowed)

Step 2: Along the sides, the T_ϕ and S values should be zero. Hence, we apply corrective forces T_L and S_L. This results in bending of the shell. Using Table 3A, find out the corrections to be made to satisfy the boundary condition. Thus, we will get corrected T_x, T_ϕ, S and M_ϕ values.

Step 3: The addition of the membrane forces from Step 1 and corrections from Step 2 give us the final results.

The procedure is the same as in Example 5.2, except that we use a different table (Table 3A) of Manual No. 31 for short shells.

Reference	*Step*	*Calculations*
	1.	*Calculation of ($\phi_k - \phi$) values for given short shell* Table 1 for membrane forces gives coefficients for $(\phi_k - \phi)$ values. However, Table 3, applicable to short shells, gives coefficients for $s/(RtL^2)^{1/4}$. Hence, we have to find the correlation between these when we are solving problems on short shells. Thus, use Table 1B (b) of Manual No. 31 or *by calculation* to find membrane forces T_x, T_ϕ and S in terms of $(\phi_k - \phi)$ in degrees. We use t, L and r to identify type of shells. In long shells, we use r/L and t/L values. In short shells, we have one quantity combining them equal to rt/L^2. Tables have been compiled in Manual No. 31 for bending analysis for 14 values of rt/L^2 ranging from 0.002 to 0.100. We will use the values of $rt/L^2 = 0.015$ for Fourier loading for $n = 1$ for our analysis (i.e. sine loading). In bending analysis of *short shells*, the points on the shell are represented by **their distance from the edge of the shell**. To represent it in non-dimensional form, we use the expression $\frac{s}{\sqrt[4]{rtL^2}}$. For membrane analysis by Table I of Manual No. 31, these portions have to be converted to ϕ degrees.
	1(a)	*Find corresponding ϕ for various $s = 0.1 \times \sqrt[4]{rtL^2}$ values.* For $s = 0.1 \times 3.673 = r\phi = 22\phi$ Hence $\phi = 0.0167$ radians $= 0.957$ radians This means for every $s = 0.1$, $\phi = 0.957$ degrees. We convert other values also to degrees as follows (see Table 1) page 69.

Reference	*Step*	*Calculations*

TABLE 1 Calculation of $\phi = (\phi_k - \phi)$ for s values for given short shell

Point No.	$\frac{s}{(rtL^2)^{1/4}}$	s(m)	ϕ degrees	$\phi_k - \phi^o$
1	0	0	0	37.85
2	0.1	0.3673	0.957	36.89
3	0.2	0.7346	1.9124	35.94
4	0.4	1.4692	3.8248	34.03
5	0.8	2.9384	7.6496	30.20
6	1.6	5.8768	15.2992	22.55
7	3.2	11.7536	30.5984	7.25
8	3.96	14.5393	37.85 − (ϕ_k)	0.0

We use these $(\phi_k - \phi)$ values for membrane analysis using Table 1 of Manual No. 31.

TABLE 2 (Example 5.3) Interpolated Membrane Coefficients from Table IB
(This interpolation is allowed when using Manual No. 31.)

$s/(RtL^2)^{1/4}$	$\phi_k - \phi$	Factors n = 1		
	(degrees)	T_x	S	T_ϕ
0	37.85	0.1598	**0.3903**	**0.7888**
0.1	36.89	0.1619	0.3818	**0.7991**
0.2	35.94	0.1640	0.3730	**0.8091**
0.4	34.02	0.1678	0.3560	**0.8280**
0.8	30.20	0.1750	0.3203	**0.8641**
1.6	22.55	0.1870	0.2439	**0.9227**
3.2	7.25	0.2009	0.0805	**0.9911**
3.96	0 (Crown)	0.2026	0	**1.0000**

Part I—Membrane Analysis

2. *Tables to be prepared*

[**Note:** The procedure is the same as in long shells Ex. 5.2. We first make separate tables for T_x, S, T_ϕ and M_ϕ forces as in Ex. 5.2 and fill in the values as the calculation proceeds.]

We provide separate tables for maximum edge values of T_x, S, T_ϕ and M_ϕ as follows: Tables 3, 4, 5 and 6) as shown in pages 71 and 72.

2(a). *Tabulate membrane forces,* at springing at $S = 0$, T_x *at* $x = L/2$

$$T_x = pr(L/r)^2 \text{ [coeff. for } (\phi_k - \phi) \text{ from Table 2 above]}$$
$$= 295 \times 22 \times (0.477)^2 \times 0.1598 = 235.97 \text{ for } s = 0$$

Enter this value in Table 3 shown on page 71 for T_x

Reference	*Step*	*Calculations*
		[**Note:** T_x is max. at crown as shown in Fig. 5.6 and decreases to the edges.]
	2(b).	*Tabulate membrane forces, S maximum at x = 0*
		$$S = pr\left(\frac{l}{r}\right) \text{ (Coeff. Table 2 above)}$$
Page 69		$S_L = 295 \times 22 \times \left(\frac{10.5}{22}\right) \times 0.3903 = 1270$ kg/m (membrane force)
		This should be made zero at edge at $\phi = 0$, as there is no member to carry it. As this will not be zero in membrane analysis, the edge value will have to be corrected by S_L, line load.
		Tabulate values as in Table 4 (on page 72).
	2(c).	*Tabulate membrane force T_ϕ at $x = L/2$*
Table 2		$T_\phi = pr \cos(\phi_k - \phi) = pr \times$ Coeff. from Table 2 above.
page 69		T_L at $s = 0 = 295 \times 22 \times 0.7888 = 5120$ kg/m
		At edge, $\phi = 0$ T_ϕ should be made zero by correction analysis.
		Tabulate these values in Table 5, page 72.
	2(d).	There is no moment M_ϕ by membrane analysis. M_ϕ for membrane analysis = 0 correction analysis produces M_ϕ.
		Part II—Correction Analysis
	3.	*Apply correction loads—(Table 3A – rt/L^2 = 0.015)*
		[**Note:** As the shell has free edges at the springing T_ϕ and S are the forces to be compensated. Hence, we apply correction loads T_L and S. This will produce further additional T_x, S and T_ϕ forces and in addition M_ϕ in the shell. The results of membrane analysis and corrections are tabulated in Tables 3 to 6 shown on pages 71, 72.]
		We proceed as follows for correction analysis
		(a) Apply T_L opposite to T_ϕ at $\phi = 0$
		Tabulate T_x, S, T_ϕ and M_ϕ produced by use of coefficients. From Table 3A.
		(b) Apply S_L opposite to S at $\phi = 0$.
		Tabulate T_x, S, T_ϕ and M_ϕ produced by use of Table 3A coefficients.
		We get the following resulting values:
		(a) For T_x value (Table 3, Example 5.3 given below) for T_x forces.
		1. T_x from membrane analysis
		2. T_x from $T_L = T_L \times$ Coeff. = 5120 × Coeff.
		3. T_x from $S = T_L \times$ Coeff. = 1220 × Coeff.

Reference	*Step*	*Calculations*
		(b) For S values (Table 4, Example 5.3) page 72
		1. S from membrane analysis
		2. S from $T_L = T_L \times$ Coeff. = 5120 × Coeff.
		3. S from S = 1210 × Coeff.
		(c) For T_ϕ values (Table 5, Example 5.3)
		1. T_ϕ from membrane analysis
		2. T_ϕ from $T_L = T_L \times$ Coeff. = 5120 × Coeff.
		3. T_ϕ from S = 1210 × Coeff.
		(d) For M_ϕ values (Table 6, Example 5.3)
		1. M_ϕ from membrane analysis = 0; (t = 0.075)
		2. M_ϕ from T_L = 5120 × t × Coeff. = 384 × Coeff.
		3. M_ϕ from S = 1210 × t × Coeff. = 91 × Coeff.
		(Ref: We have used Table IB for (n = 1) and Table 3A for (n = 1) and rt/L^2 = 0.015. For more accurate results, repeat the operations for n = 3 of Fourier load using Table IC and Tables 3A for n = 3 (page 323).

TABLE 3 (Example 5.3) Distribution of T_x Forces

Table		Values of T_x forces for different points (Values of 0.1 and 0.4 have been omitted.)					
T_x	$k = \frac{s}{\sqrt[4]{rtL^2}}$	0	0.2	0.8	1.6	3.2	3.96
	$\phi°$	0	1.21	7.65	15.3	30.6	37.85
(1)	Membrane forces, 1477 × (Coeff.)	–236	–240	–257	–274	–295	–297
(2)	Due to T_L, 5120 × (Coeff.) Table 3A	+55514	+18830	–14094	–1243	+56	–
(3)	Due to S_L 1210 × (Coeff.)	+5821	+3022	–343	–226	+36	–
	Total	61100	21612	–14694	–1743	–203	–297

Comments: In this short shell as compared to the long shell of Example 5.2, the T_x forces are low. Hence, an edge beam can be omitted or we can provide a horizontal cantilever for short shells. It will be very useful when short shells are used as storage godowns or as assembly shells.

Reference	*Step*	*Calculations*

TABLE 4 (Example 5.3) Distribution of S forces at ends

S	k	0	0.2	0.8	1.6	3.2	3.96
	ϕ	0	1.91°	7.65°	15.3°	30.6°	37.85°
(1)	Membrane forces, 3098 × (Coeff.)	–1210	–1156	–997	–756	–249	0
(2)	Due to T_L, 5120 × (Coeff.)	0	–7828	–4874	+2580	–589	0
(3)	Due to S_L	+1210	+260	–314	+71	–20	0
	Total	0	–8724	–6185	+1895	–858	0

TABLE 5 (Example 5.3) Distribution of T_ϕ forces at $L/2$

T_ϕ	k	0	0.2	0.8	1.6	3.2	3.96
	ϕ	0	1.91°	7.65°	15.3°	30.6°	37.85°
(1)	Membrane forces, 3098 × (Coeff.)	–5120	–5251	–5608	–5988	–6432	–6490
(2)	Due to T_L, 5120 × (Coeff.)	+5120	+4111	–1208	–1231	+281	
(3)	Due to S_L 1210 × (Coeff.)	0	+150	+14	–80	+18	
	Total	0	–990	–5600	–7299	–6133	–6490

TABLE 6 (Example 5.3) Distribution of M_ϕ at $L/2$

M_ϕ	k	0	0.2	0.8	1.6	3.2	3.96
	ϕ	0	1.91°	7.65°	15.3°	30.6°	37.85°
(1)	Membrane	0	0	0	0	0	0
(2)	Due to T_L, .384 × (Coeff.)	0	+49	–130	–178	+30	0
(3)	Due to S_L 91 × Coeff.	0	+3	–1	–9	+1	
	Total	0	+52	–131	–187	+31	

The final values of T_x, S, T_ϕ and M_ϕ at various points of the shell are total values in the above tables.

REVIEW QUESTIONS

1. What is meant by "membrane analysis" and "bending analysis" of shells?
2. Explain why bending analysis of shells is necessary. In what types of shell can we expect the effect of bending analysis to be less?
3. Explain how ASCE Manual No. 31 gives the tables for bending analysis of shells.
4. Are we allowed to interpolate the values of Manual No. 31? What are other publications that can supplement the values given in ASCE Manual No. 31?

REFERENCES

[1] ASCE Manual No. 31, American Society of Civil Engineers, USA, 1952. (Reference 2 below also gives Tables 2A and 2B of this manual.)

[2] Billington, D.P., *Thin Shell Structures*, McGraw-Hill, New York, 1965.

[3] Design Constants for Interior Cylindrical Concrete Shells, Advanced Engineering Bulletin, Portland Cement Association, Illinois, USA, 1960.

[4] Coefficients for Design of Cylindrical Concrete Shells Roofs, Portland Cement Association, Illinois, USA, 1959. (This is an extension of Tables 2A and 2B of Manual No. 31 for various ϕ_k, r/t and r/L values).

[5] Design of Barrel Shell Roofs, Concrete Information Series, Portland Cement Association, Illinois, USA, 1954.

[6] Ramaswamy, G.S., *Design and Construction of Shell Roofs*, McGraw Hill, New York, 1968.

[7] Chandrashekara, *Analysis of Thin Concrete Shells*, *Tata McGraw Hill*, New Delhi, 1986.

6

BEAM THEORY FOR LONG CYLINDRICAL SHELLS

6.1 INTRODUCTION TO BEAM THEORY

In beam analysis of cylindrical shells, we assume that the shell acts *as a curved beam* and we use the standard formula for longitudinal stresses and shear stresses. This theory is considered to be applicable to long shells with $L > 3r$ and specially for interior shells of multiple shell layouts. This approximate analysis should be used only when tables or computer programs are not available. Deflections calculated on this theory are only 7 percent than that of more rigorous methods and hence deflection calculations on beam theory are not reliable in all cases [1].

6.2 CYLINDRICAL SHELL ANALYSIS BY BEAM METHOD

Let us consider the cylindrical shell as a beam of section shown in Fig. 6.1.

Bending stress $= f = \dfrac{M}{I}y$ (6.1)

As we normally deal with forces,

Force $T = \left(\dfrac{M}{I}y\right)t$ (6.2)

Shear stress $= q = \dfrac{V(A \times \bar{y})}{Ib}$

and shear force $S = qt$ (6.3)

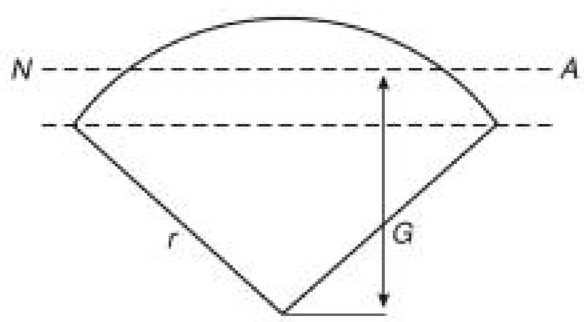

Figure 6.1 Bending of circular cylindrical shells without edge beams.

where $(A \times \bar{y})$ is the first moment of cross section up to the point of consideration about the N.A.

We can express the quantities involved in more detail as follows:

1. For the calculation of T_x for shell, we have the following items [3]:

   (a) The distance of the centre of gravity of the shell section from the centre for a shell of half centre angle ϕ_x and radius r without edge beam is given by the formula

   $$\text{Distance of C.G. of shell from centre } = G = \frac{r \sin \phi_K}{\phi_K} \tag{6.4}$$

   $$\text{Distance of the top fibre from the C.G. (or N.A.) } Z = (r - G) = r\left(1 - \frac{\sin \phi_K}{\phi_K}\right)$$

   (b) Moment of inertia of the shell about N.A. taking t as thickness of the shell,

   $$I = tr^3\left(\phi_K + \frac{\sin 2\phi_K}{2} - \frac{2\sin^2\phi_K}{\phi_K}\right) = tr^3(K) \tag{6.5}$$

   We may, however, tabulate K as given in Table 6.1 for easy use in design.

TABLE 6.1 *K for moment of inertia of cylindrical shell without edge beams*

ϕ_K° (degrees)	K
22.5	0.00041
25.0	0.00068
27.5	0.00110
30.0	0.00168
35.0	0.00358
37.5	0.00502
40.0	0.00687
45.0	0.01216
50.0	0.03174
60.0	0.04782

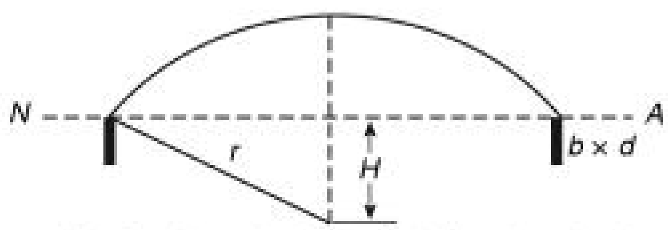

Figure 6.2 Bending of cylindrical shells with edge beams.

   (c) The centre of gravity of the shell with edge beam ($b \times d$) from centre O can be found by taking moments about the centre of the circle as in Fig. 6.2[3].

   $$\text{Distance of C.G. from centre } O = H = \frac{\left(tr^2 \sin \phi_K\right) + \left(r\cos\phi_K - \frac{d}{2}\right)(b \times d)}{\left(rt\phi_K + bd\right)} \tag{6.6}$$

   $$\text{Distance from top fibre from N.A.} = Z = r - H \tag{6.6a}$$

   (d) The moment of inertia of the shell with edge beam about the neutral axis is found as follows. We first find M.I. about centre O and then, by theory of parallel axis, determine M.I about the centre of gravity.

      (i) Moment of inertia of shell only about centre O

   $$I_{OS} = 2tr^3\left(\frac{\phi_k}{2} + \frac{\sin 2\phi_k}{4}\right) \tag{6.7}$$

(ii) Moment of inertia of *two edge beams* about centre O is,

$$I_{OB} = \frac{bd^3}{6} + 2bd\left(r\cos\phi_k - \frac{d}{2}\right)^2 \tag{6.8}$$

(iii) Total moment of inertia about centre

$$I_{OT} = I_{OS} + I_{OB} \tag{6.9}$$

(iv) Moment of inertia about neutral axis

$$I_{NA} = I_{OT} - (2rt\phi_k + 2bd)H^2 \tag{6.10}$$

where is the distance from the centre of the curve to the neutral axis.

(v) Distances of extreme fibres from N.A. are the following:

Distance of fibre at crown = Z from N.A. [Equation (6.6a)]

Distance of bottom fibre = Distance of bottom of beam from N.A.
= $(H - r\cos\phi_k) + d$ (Figure 6.2)

From these, we can calculate T_x values at crown, the valley of the shell and the bottom of edge beam as follows:

1. **Calculation of T_x.** To find maximum value of T_x at the centre of the length of the shell, we find M = bending moment assuming the shell as a beam with a w per unit length of the shell (not per m^2).

$$M = \frac{wl^2}{8} \text{ and stress } f = \left(\frac{M}{I}\right)y$$

where y = distance of fibre from the neutral axis N.A.

$$\text{Force } T_x = f \times t$$

where t is the thickness of the shell. (6.11)

[**Note**: The values of neutral axis, I and e, are only approximate as the area of steel has been neglected in our calculations.]

2. **Calculation of (Shear).** We use the well-known formula for shear distribution in a beam,

$$q = \frac{SA\bar{y}}{Ib} = \frac{VQ}{I.b} \text{ and } S = qt$$

$$S = \frac{VQ(t)}{I(2t)} = \frac{VQ}{2I} \tag{6.12}$$

where $Q = (A \times \bar{y})$, S = (Stress $\times t$)

V = Shear (maximum at ends for a beam.)

(Q at ϕ from the crown of the shell to the neutral axis should correspond to $A\bar{y}$ of the standard shear formula.)

$$Q = \left[r(2\phi)t\right] \times r\left(\frac{\sin\phi}{\phi} - \frac{\sin\phi_k}{\phi_k}\right) = 2r^2t\left(\sin\phi - \frac{\phi}{\phi_k}\sin\phi_k\right) \text{ in m}^2 \tag{6.13}$$

(See Example 6.3, step 7 where ϕ is shown as ϕ_0.)

6.3 ARCH ANALYSIS FOR CALCULATION OF T_ϕ AND M_ϕ

For the calculation of T_ϕ and M_ϕ, we need the arch analysis as shown in Figure 6.3. The arch analysis is much more involved than the beam analysis. The elastic centre method or the column analog method can be used for this purpose.

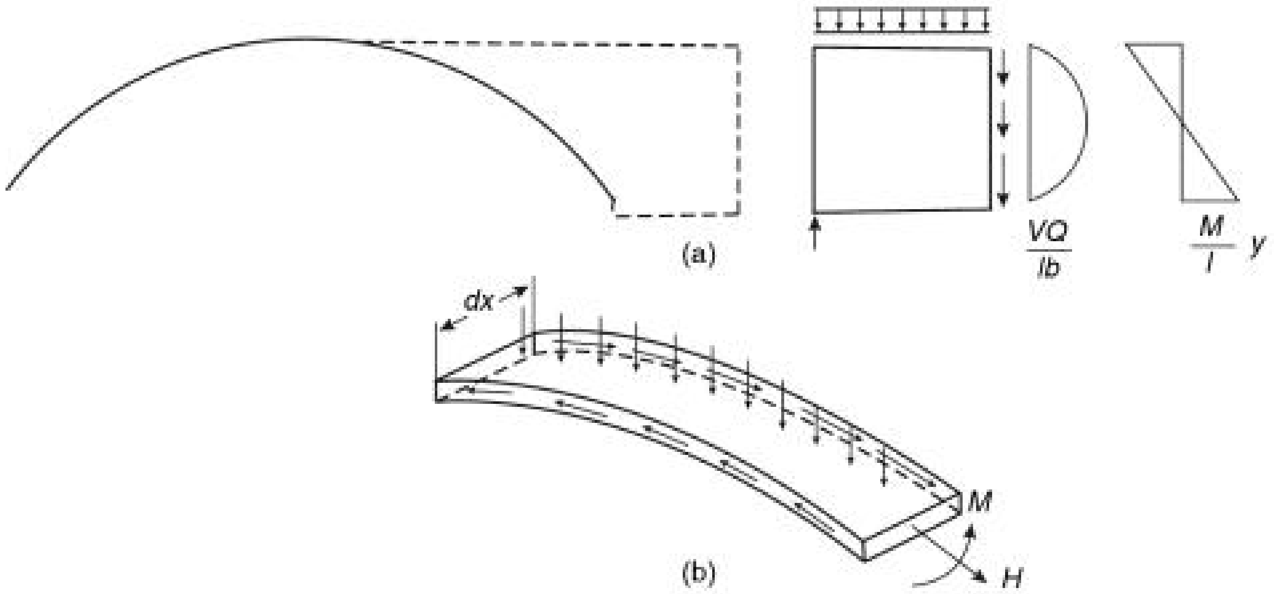

Figure 6.3 Analysis of cylindrical shells: (a) Beam analysis, and (b) Arch analysis.

6.4 TABLES FOR T_x, T_ϕ, S AND M_ϕ FOR INTERIOR SHELLS WITHOUT EDGE BEAM

Prame and Conner [3] have given us tables for the determination of T_x, S, T_ϕ and M_ϕ for dead load and live load for shells without edge beams, which are applicable to interior shells without edge beams. Some of these values are given in Table 6.2 [2] [3].

As we have already noted in Section 5.5.1 that the dead load gives greater stresses in the shell than the same live load, we may either add the total live load (which is usually small) to dead load so that the results are on the safe side. Alternatively, we may convert live load to equivalent dead load by the formula $p' = p(\sin \phi_K)/\phi_K$ so that we need to use only the tables for dead load in our calculation.

TABLE 6.2 Coefficients for beam analysis of circular cylindrical shells without edge beams

$$T_x = \frac{L^2}{r}(p \times \text{Coeff.}) \quad \text{and} \quad T_\phi = r(p \times \text{Coeff.})$$

$$S = -L(p \times \text{Coeff.}) \quad \text{and} \quad M_\phi = r^2(p \times \text{Coeff.})$$

where p = dead load on the shell per unit area. (As an approximation, we convert live load into an equivalent dead load explained in Section 5.5.1.)

TABLE 6.2 Coefficients for Beam Analysis of Cylindrical Shells [1]

ϕ_K (deg.)	ϕ	Values of coefficients for dead loads			
		$T_x(1)$	$T_\phi(2)$	$S(3)$	$M_\phi(4)$
22.5	ϕ_K(crown)	–6.167	–1.433	0.000	–0.00309
	0.75 ϕ_K	–5.003	–1.205	2.269	–0.00118
	0.50 ϕ_K	–1.521	–0.615	3.626	0.00245
	0.25 ϕ_K	4.245	0.065	3.165	0.00249
	0 (springing)	12.239	0.384	0.000	–0.00702
25.0	ϕ_K	–5.012	–1.430	0.000	–0.00378
	0.75 ϕ_K	–4.064	–1.202	2.049	–0.00145
	0.50 ϕ_K	–1.232	–0.613	3.273	0.00300
	0.25 ϕ_K	3.451	0.064	2.855	0.00304
	0	9.920	0.374	0.000	–0.00857
27.5	ϕ_K	–4.158	–1.426	0.000	–0.00453
	0.75 ϕ_K	–3.370	–1.199	1.869	–0.00173
	0.50 ϕ_K	–1.018	–0.610	2.985	0.00360
	0.25 ϕ_K	2.863	0.063	2.602	0.00363
	0	8.220	0.363	0.000	–0.01025
30.0	ϕ_K	–3.508	–1.422	0.000	–0.00533
	0.75 ϕ_K	–2.842	–1.195	1.720	–0.00203
	0.50 ϕ_K	–0.856	–0.607	2.746	0.00424
	0.25 ϕ_K	2.417	0.061	2.392	0.00426
	0	6.920	0.352	0.000	–0.01204
32.5	ϕ_K	–3.002	–1.418	0.000	–0.00618
	0.75 ϕ_K	–2.431	–1.191	1.595	–0.00235
	0.50 ϕ_K	–0.729	–0.603	2.544	0.00235
	0.25 ϕ_K	2.069	0.060	2.215	0.00492
	0	5.908	0.339	0.000	–0.01393
35.0	ϕ_K (crown)	–2.601	–1.414	0.000	–0.00707
	0.75 ϕ_K	–2.105	–1.186	1.488	–0.00268
	0.50 ϕ_K	0.629	–0.600	2.372	0.00565
	0.25 ϕ_K	1.793	0.058	2.064	0.00561
	0 (valley)	5.105	0.326	0.000	–0.01591
37.5	ϕ_K (crown)	–2.278	–1.409	0.000	–0.00800
	0.75 ϕ_K	–1.842	–1.181	1.396	–0.00302
	0.50 ϕ_K	–0.548	–0.596	2.224	0.00640
	0.25 ϕ_K	1.571	0.057	1.933	0.00632
	0 (valley)	4.458	0.312	0.000	–0.01796
40.0	ϕ_K (crown)	–2.013	–1.404	0.000	–0.00897
	0.75 ϕ_K	–1.627	–1.176	1.315	–0.00337
	0.50 ϕ_K	–0.482	–0.592	2.095	0.00719
	0.25 ϕ_K	1.389	0.055	1.819	0.00705
	0 (valley)	3.928	0.297	0.000	–0.02006

6.5 ANALYSIS OF NORTHLIGHT CYLINDRICAL SHELLS

Northlight shells are specially suitable in factories where good lighting for precision work is needed. These shells are frequently used for spans of 12 to 15 m without prestress. With prestress, spans up to 45 m are possible. As they are long shells, beam theory can be used for their design. The following are the recommended dimensions with reference to Figure 6.4:

1. Radius is to be less than 12 m. Central angle is of the order of 60 degrees.
2. Normal ratio of width to span is between 1:2 to 1:4.
3. The height h from its base should not be less than 1/6 span.
4. The height of circular shell h from its chord should be greater than $L/18$.
5. The height of gutter edge beam should not be less than $L/18$.

When there are a series of these shells, as in a factory, a number of them may be supported at their ends on a single transverse or stiffener. As the shell is not symmetric about the vertical axis, special procedure is required for its analysis by beam method. Reference [3] gives details of the analysis of the shell. Reference [4] gives details of design and tables for easy design of northlight cylindrical shells. These can be used in actual designs.

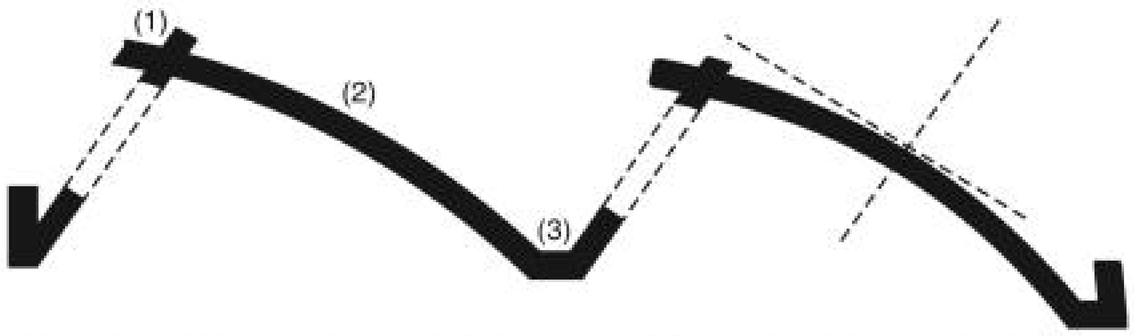

Figure 6.4 Northlight circular cylindrical shells: (1) Crwon beam, (2) Circular shell, and (3) Valley or gutter beam.

SUMMARY

The beam method of design of cylindrical shells can give an approximate solution when applied to long shells. For these shells, it can be used to check the output of more theoretical method of design by use of tables or by computer method. However, it is not applicable to design of short shells where arch action is more predominant than beam action.

EXAMPLE 6.1 [Beam analysis of long shell ($L = 3r$) without edge beam by Table 6.1] Determine T_x, T_ϕ, S and M_ϕ forces on a shell with L = 36 m, r = 9 m, t = 75 mm and $\phi_K = 40°$. There is no edge beam for the shell. The live load be taken as 0.8 kN/m^2 [4] (see Figure 6.1).

Reference	*Step*	*Calculations*
		Part I—Finding T_x by Calculation
	1.	*Check dimensions of the shell* $L/r = 36/9 = 4 > \pi$ or 3 It is a long shell. Beam theory is applicable.
	2.	*Find loading per metre length of shell* $p(\text{DL}) = 0.075 \times 24 = 1.8\ \text{kN/m}^2$ Live load $= 0.8\ \text{kN/m}^2$; Equivalent DL $= 0.8 \times \sin\phi_K/\phi_K = 0.73\ \text{kN/m}^2$ Design for total dead load equal to $2.5\ \text{kN/m}^2$ Total load per unit length as a bean $= pr\phi$ $= 2.5 \times 9(0.697 \times 2) = 31.4\ \text{kN/m}$
	3.	*Find N.A. from centre* $G = \dfrac{r \sin\phi_K}{\phi_K} = \dfrac{9 \times \sin 40}{0.697} = 8.3\ \text{m}$ and $Z = (r - G) = 9 - 8.3 = 0.7\ \text{m}$ Depth of springing $= G - \cos 40 = 1.4\ \text{m}$
Table 6.1	4.	*Calculate I* $I = tr^3 K = 0.075 \times (9)^3 \times 0.00687 = 0.3756$
	5.	*Find max. value of T_x with $M = wl^2/8$* $M_{\max} = \dfrac{31.4 \times 36 \times 36}{8} = 5087\ \text{kNm}$ and $T_x\left(\dfrac{M}{I} \times y\right)(t)$ T_x at crown $= \left(\dfrac{5087 \times 0.7}{0.3756}\right)(0.075) = 711$ kN/m (compression) T_x at springing $= \dfrac{M}{I}(y) \times t = \dfrac{5087 \times (1.4) \times 0.075}{0.3756} = 1422$ kN/m (tension) [**Note:** Usually, the beam method underestimates the values at springing and overestimates value at the top. By exact analysis, we getat T_x at top = 626.6 kN (comp.) against 711 kN/m as above T_x at bottom = 1884.5 kN (tension) against 1422 kN/m as above [We may also calculate shear S from Eqs. (6.12) and (6.13)]
		Part II—Use of Table 6.2 instead of calculations
Table 6.2	5(a).	*Find value of T_x by Table 6.2 instead of calculation ϕ_k =* T_x at crown $= L^2/r$ ($p \times$ Coeff. in Table 6.2) $= \dfrac{36 \times 36}{9}(2.5 \times 2.013) = 724\ \text{kN/m}$

Reference	*Step*	*Calculations*
		(This is comparable to 711 kN/m we get in step 5)
		T_x at springing $= \dfrac{(36)^2}{9}(2.5 \times 3.928)$
		$= 1414$ kN/m (compared to 1422 in step 5)
	6.	*Calculate T_ϕ by use of Table 6.2*
		Max. T_ϕ at centre; at $\phi = \phi_K$ crown $= rp$ (Coeff. in Table 6.2)
		$T_\phi = 9 \times 2.5(-1.404) = 31.6$ kN (comp.)
		At $\phi = 0$ springing $T_\phi = 9 \times 2.5(0.297) = 6.7$ kN
	7.	*Shear S at $\phi = 20°$ at support (max)*
		$S = -L \times p(\text{Coeff.}) = -36 \times 2.5(2.095) = -188$ kN/m
		(Considered +ve where it creates tension in the direction of increasing values of x and ϕ)
	8.	*Calculate moment M_ϕ @ crown and springing*
		M_ϕ @ crown $= r^3\ (p)$ (Coeff.)
		$= (9)^3 \times 2.5(-0.00897) = -16.35$ kNm
		(–ve means tension in outer fibre or on the top of the shell)
		M_ϕ at springing $= (9)^3 \times (2.5)(-0.002006) = -36.6$ kN (tension in the outer fibre)

EXAMPLE 6.2 [Beam Analysis of shell without edge beam]

Estimate T_x value of barrel shell of Example 5.1 with $L = 15$ m, $r = 7.5$, $t = 75$ mm, $w = (230$ – no edge beam) by the beam theory for barrel shells.

(L/r value = 15/7.5 = 2 (less than 3). In this case, analysis is strictly not applicable.)

Reference	*Step*	*Calculations*
	1.	*Find M.I. of shell*
		C.G. at H from $O = \dfrac{R \sin \phi_k}{\phi_k} = \dfrac{7.5 \times \sin 40}{0.6977} = 6.91$ m
Table 6.1		$y_{\text{top}} = 7.5 - 6.91 = 0.6$ m
		$y_{\text{bottom}} = \text{rise} - y_t = 1.755 - 0.6 = 1.155$ m
		I by formula for $\phi_k = 40 = 0.00687\ R^3 t$
		$I = 0.00687 \times (7.5)^3 \times 0.075 = 0.2174$ m^3
	2.	*Estimate value of T_x*
		Load per m of shell $w = (2\pi \times 7.5 \times 225 \times 80)/360 = 2354$ kg/m
		Stress at bottom $= \left(\dfrac{2354 \times 15 \times 15}{8}\right)\left(\dfrac{1.155}{0.2174}\right) = 350 \times 10^3$ kg/m

Reference	*Step*	*Calculations*
Chapter 5		$T_x = f \times t = 350 \times 10^3 \times 0.075 = 26250$ kg (tension)
		From Example 5.1, value of T_x at valley = 63759 kg (tension)
		T_x at top = 26250 × (0.6/1.155) = 13636 kg/m (comp.)
		From Example 5.1, the value = 893 (comp.)
		[**Note:** These values are different from the exact solution got in Example 5.1. However, if we plot the distribution of stress across the whole section, it can be seen that the area of tension zone by the approximate beam theory is larger than by the exact theory of shells. Hence, the total tension forces for which steel has to be provided will not be much different in the two solutions. Hence, both solutions may require the same amount of steel reinforcement.]

EXAMPLE 6.3 (Analysis of a long shell with edge beam by beam theory)

The dimensions of a shell are the following [Refer Example 8.3]

$L = 25$ m; $\phi_k = 35°$; $r = 7.5$ m; $t = 75$ mm. Edge beam size is 20 cm × 150 cm.

The shell load with waterproofing can be assumed as 370 kg/cm^2. (This is the sample Example 5.3 worked out in Chapter 8 by the ASCE Manual method.)

Reference	*Step*	*Calculations*
	1.	*Find total wt. per* m *of shell and edge beams*
		Weight of 2 beams = 2(0.25 × 1.5 × 2400) = 1800 kg/m
		Shell wt. = 2 × $(r\phi_k) \times w$ = 2 × 7.5 × 0.6 × 1370 = 3386 kg/m
		Total = 5186 kg/m = w
	2.	*Find C.G. of shell with edge beams*
Eq. (6.6)		$H \text{ from } O = \dfrac{tr^2 \sin\phi_k + \left(r\cos\phi_k - d/2\right)(b \times d)}{\left(rt\phi_k + bd\right)}$
		$H = \dfrac{\left(0.075 \times (7.5)^2 \times \sin 35\right) + (7.5\cos 35 - 0.75)(1.5 \times 0.025)}{(7.5 \times 0.075 \times 0.61) + (1.5 \times 0.025)}$
		$H = \dfrac{2.419 + 2.02}{0.343 + 0.375} = 6.18 \text{ m}$; $Z = 7.5 - 6.18 = 1.32$ m
		Dist. to beam bottom = $(H - 7.5 \cos \phi_k) + d$ = 1.54 m = 154 cm
		(Hence, C.G. nearly at top of the beams.)
	3.	*Find I of shell about centre O*
Eq. (6.7)		$I_{OS} = 2tr^3\left(\dfrac{\phi_k}{2} + \dfrac{\sin 2\phi_k}{4}\right)$
		$= 2 \times 0.075 \times (7.5)^3(0.305 + 0.235) = 34.14 \text{ m}^4$

Reference	*Step*	*Calculations*
	4.	*Find I of two beams about centre O and total I_0*
Eq. (6.8)		$I_{OB} = \dfrac{2bd^3}{12} + 2bd\left(r\cos\phi_k - \dfrac{d}{2}\right)^2$
		$= \left[\dfrac{0.25\times(1.5)^3}{6}\right] + [2\times0.25\times1.5](7.5\cos35 - 0.75)^2$
		$= (0.14 + 21.82) = 21.96$ m^4
Eq. (6.9)		Total $I_0 = 34.14 + 21.96 = 56.10$ m^4
	5.	*Find I about C.G. (N.A.)*
		$I = I_0 - (2Rt\phi_k + 2bd) \times H^2$
		$= I_0 - [(2 \times 7.5 \times 0.075 \times 0.61) + 0.75](6.18)^2$
		$= 56.1 - 54.8 = 1.3$ m^4
	6.	*Find max. compression and tension in shell taking shell as curved beam*
		$M = \dfrac{wL^2}{8}$ and $f = -\dfrac{M}{I}y$
		1. Max. comp. in shell concrete $= \dfrac{5186\times25\times25\times(1.32)}{8\times1.3}$
		$= 411389$ kg/m^2 = 41 kg/mm^2 = 4N/mm^2
		(This compression is allowable in M20 concrete.)
		2. Max. tension $= \dfrac{5186\times25\times25\times(1.54)}{8\times1.3}$
		$= 479954$ kg/m^2 = 48 kg/cm^2
		[**Note:** This value is comparable with the value got in Example 8.2 as 50.4 kg/cm^2 by exact analysis in Chapter 8.]
	7.	*Find total tension for reinforcement*
		Let position of N.A. at ϕ_0
		Find ϕ_0
Step 2		$7.5 \cos \phi_0 = 7.5 - 1.32 = 6.18$
		Hence $\phi_0 = 34.5°$
		N.A. is nearly on top of the beam. [Area in tension = 20 × 150 cm^2]
		Tension on bottom fibre = 48 kg/cm^2
		Total tension = $\dfrac{1}{2}$ 48 × 20 × 150 = 72,000 kg
		(This is quite in agreement with 87,682 kg got from exact analysis in Example 8.3, step 10, Chapter 8.)

Reference	*Step*	*Calculations*
	8.	*Calculation of shear force S* Using Eqs. (6.12) and (6.13), the value of the shear force S can also be calculated. [**Note:** As the shell is a curved surface, calculations of shear based on linearity of stress will yield higher results for compression.] **[Comments: The membrane analysis can be used for the calculation of forces for preliminary design of the central part of short sells, and the beam method can be used for the calculation of the T_x forces at the edges for shells without edge beam and also for preliminary design of edge beams of long shells.]**

REVIEW QUESTIONS

1. What is meant by "beam method" of analysis of cylindrical shells? For what types of shells is this method applicable?
2. Explain why the beam method of analysis is not suitable for short shells. What approximate method can be used for these shells?
3. Sketch the layout of a series of northlight cylindrical shells for a factory. Give the dimensions you will adopt for the layout of a series of shells of 18 metre spans.

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7

STATIC CHECKS OF RESULTS OF ANALYSIS OF CYLINDRICAL SHELLS

7.1 INTRODUCTION

In this chapter, we deal with the statical checks that can be made on the output of the theoretical analysis.

7.2 STATICAL CHECK USING LAWS OF EQUILIBRIUM

There are three statical checks we can make, using the results of the analysis[1]. They are:

1. At the centre of the shell at $x = L/2$. The total tension acting on the section should be equal to the total compression $\Sigma T_x = 0$.
2. At support (at one end $x = L$), Σ vertical component of shear = $V = 1/2$ [Load on shell].
3. At the centre of the shell, the total external moment should be equal to the internal moment of T_x forces in the section.

7.2.1 Check No. 1 Checking ΣT_x forces $= 0$ at $x = L/2$

This check simply means that, as in a beam, in any section total tension should be equal to compression. To sum up the forces, we use Simpson's rule with odd number of ordinates. We use one half of the section and choose values at, say, 10° interval of the shell.

$$T = \frac{\Delta L}{3}[y_0 + \Sigma(4 \times \text{Odd ordinates} + 2 \times \text{Even ordinates}) + y_n] \tag{7.1}$$

For 10 degree intervals, $\Delta L = \frac{2\pi R}{360} \times 10°$. This check is illustrated by Example 7.1.

[**Note:** It is interesting to note that when $(\phi_k - \phi) = 90°$, we have both T_x and T_ϕ equal to zero at the edges. Hence, if we cut away the lowest half of a full cylinder $\phi_k = 90°$, the shell will carry its own weight between the two edges. However, shear S is not zero at the springing, so at the support, considerable tensile stresses will have to be developed. (See Example 5.1.)]

7.2.2 Check No. 2 Checking Shear at Support

Take vertical component of the shear as shown in Eq. (7.2).

$$V = \frac{W}{2} = \frac{\Delta L}{3}\left[V_1 + 4\Sigma\,\text{Even} + 2\Sigma\,\text{Odd} + V_n\right] \tag{7.2}$$

where ΔL is as described in Sec. 7.2.1.

$$\text{Total shear at support (with sine loading)} = \frac{1}{2}(\text{Total load}) = \frac{4L}{\pi^2}(\text{UDL})$$

$$\Sigma \text{ Vertical component of } S \text{ at } X = 0 \text{ must be } = \frac{1}{2}(\text{load on shell})$$

Note that only vertical components as in Eq. 7.1 should be considered, i.e., $S \times \sin(\phi_k - \phi)$.

7.2.3 Check No. 3 Checking Moment at $L/2$ for Forces

As there are no horizontal forces acting on the shell, the moment of T_x forces at any point at $x = L/2$ must be equal to the external moment at $x = L/2$. If we take moments about the bottom edge, the lever arm for a point ϕ from the edge will be,

$$\text{Lever arm} = R[\cos(\phi_k - \phi) - \cos\phi_k] = z$$

Again, using Simpson's rule, we get

$$M = 2 \times \frac{\Delta L}{3}[y_0 + 4y_1 + 2y_2 + \cdots + y_n]$$

$$\text{Moment} = \frac{wl^2}{8} \text{ for sine load} = \left(\frac{4}{\pi^3}\right)wL^2$$

$$\text{The sum of the moment of } T_x \text{ forces about bottom edge} = \frac{4}{\pi^3}\left(wL^2\right) = \frac{wL^2}{7.74}$$

where w = load per metre length. (This is the moment of the sine loading.)

[**Note:** The T_ϕ force for design purposes can be assumed to be uniform in the longitudinal direction, even though theoretically it decreases to zero at the support. The transition from zero to full value, however, takes place over a very short interval. Thus, especially for values near the crown, the assumption of uniform distribution of T_ϕ due to a UDL along the length can be justified.]

SUMMARY

This chapter explains the static checks that can be made to verify the results of analysis of long cylindrical shells. The checks are similar to those which apply for a beams.

EXAMPLE 7.1 Make statistical check for $\Sigma T_x = 0$ on the results of Example 6.1 $\phi_k = 40°$.

Reference	*Step*	*Calculations*
	1.	*Find* ΔL $\Delta L = \frac{2\pi r}{360} \times 10 = \frac{2 \times 3.16 \times 7.5 \times 10}{360} = 1.308$ m ΣT_x = Twice × $\left(\frac{\Delta L}{3}\right)$(First + $4\Sigma y_1$ + $2\Sigma y_2$ + Last) on each half (with odd number of ordinates) Values of T_x are the following: $(\phi_k - \phi)$ 40 30 20 10 0 T_x 63579 –2230 –14117 –6161 –893 Considering each half of shell, we have to multiply one side by 2. $\Sigma T_x = \frac{2 \times 11308}{3}[63579 + (4 \times 2230) - 2(14117) - 4(6161) - 893]$ $= K[63579 - 8920 - 28234 - 24644 - 893]$ $= K[63579 - 62691] = 774$ Difference = $\left(\frac{774}{63579}\right) \times 100$ = 1.2% only

EXAMPLE 7.2 From data of Example 7.1 make a statical check for ΣS at end of the shell. It must be equal to end reaction (shear).

Reference	*Step*	*Calculations*
	1.	*The following are the shear values at every* 10° *in Example 6.1:* $(\phi_{LI} - \phi)$ 0 10 20 30 40 S 0 1304 3774 6572 0 ΔL for 10° = $\frac{2 \times \pi \times 7.5 \times 10}{360}$ = 1.308 m
	2.	*Find the sum of the vertical components* Taking the full length of the arch, i.e. twice the value, $S = \frac{\Delta L}{3}(y_0 + 4y_1 + 2y_2 + \ldots y_n)$ [Eq. (7.2)] $S = 2\left[\frac{1.308}{3}\right][0 + 4(1304 \times \sin 10) + 2(3774 \times \sin 20)$ $+ 4(6752 \times \sin 30) + 0$ $= 0.872(906 + 2581 + 13504) = 14{,}816$ kg (a)

Reference	*Step*	*Calculations*
	3.	*Find total shear at support*
		$w(\text{Dead}) = \left[\dfrac{230 \times (2\pi \times 7.5) \times 80}{360}\right] = 2407 \text{ kg/m}$ (b)
		$\text{End reaction} = \dfrac{4L}{\pi^2}(\text{UDL}) = \dfrac{4 \times 15}{\pi^2} \times 2407 = 14{,}647 \text{ kg}$ (2)
	4.	*Equate steps 2 and 3*
		Hence, ΣS = End reaction as (a) = (b)

EXAMPLE 7.3 Using Example 6.1, make the moment check for the values got by membrane analysis.

Reference	*Step*	*Calculations*
	1.	*Values of forces at 10° interval*
		$(\phi_K - \phi)$ 0 10 20 30 40
		T_x –893 –6161 –14117 –2230 +63579
		$L.A.$ 1.75 1.64 1.3 0.75 0
		(Lower arm is calculated from equation
Ex. 7.2		$L.A. = 7.5[\cos - \cos \phi_K]$
		For $(\phi = \phi_K) = 7.5(1 - \cos 40) = 1.75$
		Simpsons rule $= \dfrac{\Delta L}{3}[y_0 + 4y_1 + 2y_2 + \cdots + y_n]$
		$= \dfrac{2\Delta L}{3} = 0.872$
	2.	*Sum of internal moments*
		$M_1 = (0.872)[0 + (4 \times 0.75 \times 2230) + (2 \times 1.3 \times 14117)]$ $+ (4 \times 1.64 \times 6161) + (1.75 \times 893)$
		$= 74445$ kgm
	3.	*External moment at centre of span*
		Wt. per metre length = 2407 kg/m
		M_2 for sine load $= \dfrac{4}{\pi^3}(wL^2) = \dfrac{4}{\pi^3} \times 2407 \times (15)^2 = 69973$ kgm
	4.	*Error between M_1 and M_2*
		$= \dfrac{4472}{69973} \times 100 = 6.4\%$

REVIEW QUESTIONS

1. Explain the three checks that we can make for the results of the analysis of circular cylindrical shells.
2. To which type of shells we can apply these checks?
3. Explain how if $\phi_k = 90°$, the shell will carry its own weight.

REFERENCE

[1] Design of Cylindrical Shell Roofs, ASCE Manual No. 31, American Society of Civil Engineers, Chicago, 1952.

8

ANALYSIS OF CIRCULAR CYLINDRICAL SHELLS WITH EDGE BEAMS

8.1 INTRODUCTION

The design of edge beams is a very important component in cylindrical shells. It was shown in Example 5.1 and we can also see from the beam theory of shells that in long cylindrical shells, the longitudinal tension at the edges will be very large. It is advisable to provide an edge beam to accommodate this large tension by a vertical shallow edge beam. This beam is usually only about 1/12 to 1/16 of span in depth so that it acts as a shallow beam.

In short shells, on the other hand, as can be seen in Example 5.2, the T_x forces are not very predominant. Hence, we may provide the edge beam as a horizontal projection as shown in Figure 8.1. It will also serve as a platform or sunshade at the edge of the large span shell. In this chapter, we deal with the design of the vertical edge beams for long shells in detail and the horizontal edge beams for short shells in brief.

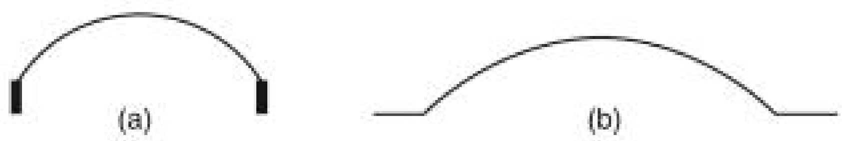

Figure 8.1 Edge beams for circular cylindrical shells: (a) Vertical edge beam for long shells, and (b) Horizontal edge beam for short shells.

In this chapter, we examine the principles of analysis of the following types of shells:

1. Single long shells with reinforced concrete vertical edge beams
2. Short shells with horizontal edge beams
3. Shells with prestressed edge beams
4. Multiple barrel shells with no edge beams
5. Multiple barrel shells with edge beams
6. Continuous shells
7. Long shell with small edge beams resting on walls or closely spaced columns.

8.2 THEORY OF THE METHOD OF ANALYSIS OF SINGLE LONG SHELL WITH R.C. EDGE BEAMS BY ASCE MANUAL NO. 31

According to the analysis recommended in Manual No. 31, we first consider the main shell and the edge beam as two different members with a common junction. At the junction, they are under the action of equal and opposite interactive forces. Only two interactive forces (vertical load and shear) are taken as indeterminates. The two conditions to be satisfied for finding the indeterminates are that the stresses and the displacements at the junction of the shell and the beam should be equal.

In general, the four conditions for the continuity of the shell and the beam can be assumed to be the following:

1. Vertical displacements should be the same.
2. The value of T_x/t at the edge of the shell and the stress in the upper edge of the beam f should be the same.
3. The horizontal displacements should be the same. These can be considered negligible.
4. The angular rotation of the common edges is the same and this also is considered as negligible.

In theory, these four conditions will reduce to four simultaneous equations. If we neglect two of them, we reduce this to two equations with two unknowns.

By choosing a shallow beam (not a deep beam), we can assume that the horizontal and rotational stiffness of that beam is negligible such that the last two conditions can be omitted. As it is easy to solve a set of simultaneous equations, we take two important unknowns and evolve conditions for solution. *In this case, we take the vertical reaction of shell due to the beam V_b and the shear at the junction S_b as the two unknowns.* Considering only two conditions of continuity to be fulfilled, we can solve for V_b and S_b. From this, we can find the stresses in the shell and the edge beams. The vertical load V_b and shear S_b are shown in Figure 8.2. For eases in analysis, as in the case of other forces in the shell, these forces are also assumed to be sinusoidal as represented by Fourier series.

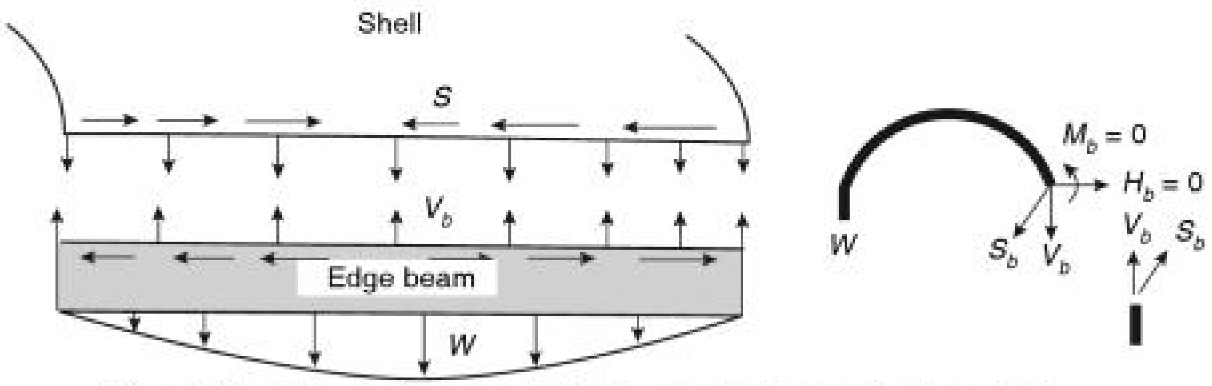

Figure 8.2 Interaction between shell and vertical edge beam for long shells.

Thus, the downward force on the beam per metre length will be $\left(\frac{4}{\pi}W - V_b\right)$, where W is the weight of the beam per metre length in sinusoidal terms. Equal and opposite vertical load V_b and shear S will act between the shell and the beam.

8.2.1 Formulae for Stresses and Deflection of Edge Beam under Vertical Load and Shear

[**Note:** We may go directly to Eqs (8.5) and (8.6). The following derivations of the formulae are given for completeness only.]

We will first analyze the edge beam subjected to a vertical load $(W - V_b) = V$ and then due to shear S at the junction of the shell and the edge beam. Let the dimensions of the beam be $(b \times h)$.

Step 1 Find the deflection and stress at top of the beam due to vertical loads $(4/\pi W - V_b) = V$ on the edge beam.

(a) *Deflection due to V (Vertical load)*

Let the size of the beam be $(b \times h)$ and length L. We first find the effect of vertical load.

We write,

$$EI\frac{d^4y}{dx^4} = V\sin\left(\frac{\pi x}{L}\right)$$

By integration, we find $y = \delta$ as

$$\delta = \frac{V}{EI}\left(\frac{L}{\pi}\right)^4 \sin\left(\frac{\pi x}{L}\right) \tag{a}$$

Putting $I = \dfrac{bh^3}{12}$ and $\dfrac{12}{\pi^4} = 0.12319$, we get

$$\delta = \frac{VL^4}{bh^3E}(0.12319)\sin\left(\frac{\pi x}{L}\right) \tag{8.1}$$

(b) *Stress at top of beam due to V*

Now, $M = EI\dfrac{dy}{dx}$. Hence by integrating Eq. (a), we can get bending moment M.

Stress $f = \dfrac{M}{Z}$

Again, we get the stress on the top fibre of beam by putting $\dfrac{6}{\pi^2} = 0.60793$,

$$f = \frac{M}{Z} = V\left(\frac{L}{h}\right)^2\left(\frac{1}{b}\right)0.60793\sin\left(\frac{\pi x}{L}\right) \tag{8.2}$$

(With compression at the top, we have –ve sign.)

Step 2 Similarly, we have to find deflection and stress in the edge beam due to shear S_b of the shell at the junction of the shell and the beam.

(a) *Stresses at top and bottom of beam*

Put shear load as $S_b \cos \pi x/L$. It is tension on top of the beam as shear increases towards the ends. Total tension is T, maximum at $x = 0$ or L. At any distance from the beam support,

$$\text{Total tension } T = \int_0^x S_b \cos\frac{\pi x}{L} dx = \frac{L}{\pi} S_b \sin\left(\frac{\pi x}{L}\right)$$

This shear acting on top surface of the beam produces a moment about the C.G. of the beam besides producing tension on top. Applying equal and opposite forces at C.G. of the beam, we get the value of moment as,

$$M = -\left[\frac{L}{\pi} S_b \left(\frac{h}{2}\right)\right] \sin\frac{\pi x}{L} \tag{b}$$

The stress due to the combined effects of T and M on the top due to S_b is,

$$f_L = \frac{T}{A} + \frac{M}{Z} = S_b \left(\frac{L}{\pi bh} + \frac{3L}{\pi bh}\right) \sin\frac{\pi x}{L} = \left(\frac{L}{bh}\right) 1.2732 S_b \sin\frac{\pi x}{L} \tag{8.3}$$

The stress at the lower edge due to the tension T and moment M,

$$f_b = \left(\frac{T}{A} - \frac{M}{Z}\right) = -\left(\frac{L}{bh}\right) 0.6366 S_b \sin\frac{\pi x}{L} \tag{8.3a}$$

(This is only one half of f_{top} and the compression is –ve in sign.)

(b) *Deflection*

δ is got by integrating $\dfrac{EId^2y}{dx^2} = -M$. Integrating twice, we get,

$$\delta = -\left(\frac{6}{\pi^3}\right)\left(\frac{L^3}{bh^2E}\right) S_b \sin\frac{\pi x}{L} = -\left(\frac{L^3}{bh^2E}\right)(0.19351) S_b \sin\frac{\pi x}{L} \tag{8.4}$$

Step 3 Find the total effect of vertical load and shear in the beam on the edge beam.

Adding the effects of $(4/\pi)(W - V_b)$ and S_b, we get the final equations for stresses [Eq. (8.2) + Eq. (8.3)] and deflection of beam [Eq. (8.1) + Eq. (8.4)] as,

Stress in beam = f_{top}

$$f_{\text{top}} = -\left(\frac{L}{h}\right)^2 \left(\frac{1}{b}\right)\left[0.60793\left(\frac{4}{\pi}W - V_b\right) + 1.2732\left(\frac{L}{bh}\right) S_b\right] \sin\frac{\pi x}{L} \tag{8.5}$$

Deflection of beam = δ

$$\delta = \frac{L^4}{bh^3E}\left[0.12319\left(\frac{4}{\pi}W - V_b\right) - 0.19351\left(\frac{L}{bh^2E}\right) S_b\right] \sin\frac{\pi x}{L} \tag{8.6}$$

Equations (8.5) and (8.6) are *important results* and we will use these "ready to use equations" in the design of shells with edge beams.

[**Note:** The following formulae for stresses and deflection for the shell part in contrast require more calculations from fundamentals.]

8.2.2 Formulae for Deflection and Stresses of Shell

In the previous section, we derived the stresses and deflection in the beam. Next, we consider the stresses and deflection of the shell under the various line loads, especially V_b and S_b. We consider the shell separately from the beam. As we have already studied the theory of the analysis of the shell in Chapter 5, here we only enumerate the steps to be followed.

Step 1 Analysis for T_x forces in the shell due to shell load and V_b and S_b

We first analyze the shell for *the edge value of* T_x under the *action of the load by membrane analysis* and then also due to V_b and S_b. This is carried out in the following ways:

(a) By membrane analysis to find T_x, S and T_ϕ.
(b) Find T_x due to corrective forces V_L, H_L and S. (These two steps give a resultant T_x.)
(c) Find T_x due to V_b and S_b from beam.

The stress in the shell due to the total value T_x will be T_x/t (where t is thickness of the shell). The form of the value will be as follows (the letter F denoting "a function of"). The value of each T_x is formed by using Table 2A of Manual No. 31.

$$\text{Stress} = \frac{T_x}{t} = \left(\frac{L}{r}\right)^2 \left(\frac{1}{t}\right)\left[\text{Constant} + F(V_b) + F(S_b)\right]\sin\frac{\pi x}{L}$$

Step 2 Find the displacement of the shell Δ due to V_b and S_b

1. Due to membrane displacement due to V_L, H_L and S in the shell using Table 1B.
2. Due to V_b and S_b, i.e. reactions from beam by use of Table 2B.

Thus by using Tables IB and Table 2B, the total deflection Δ is found out. (This is illustrated in Example 8.3.)

The form of the stress value and deflection at the edge of the shell will be as follows: (F denotes function of.)

Longitudinal stress in shell = $\dfrac{T_x}{t}$.

$$\frac{T_x}{t} = \left(\frac{L}{r}\right)^2 \left(\frac{1}{t}\right)[\text{Constant} + F(V_b) + F(S_b)]\sin\frac{\pi x}{L} \quad \text{(stress in shell)} \tag{8.7}$$

Vertical deflection of shell ΔV,

$$\Delta = \frac{L^4}{r^3 tE}\left[\text{Constant} + F(V_b) + F(S_b)\right]\sin\frac{\pi x}{L} \quad \text{(deflection in shell)} \tag{8.8}$$

Equations (8.7) and (8.8) are important equations we use for edge beam analysis. (These are not ready-to-use equations as each term has to be calculated by using ASCE Manual tables.)

8.2.3 Summary of Final Conditions to be Satisfied

The only unknowns are V_b and S_b. We proceed as follows to find them:

Step 1 Write down first conditions of compatibility of equality of stresses [Equate Equations (8.5) to (8.7)] in beam and shell.

The following are the formulae for expressing the compatibility of stresses between the shell and the edge beam:

Equation 8.5; $f_{\text{top}} = -\left(\frac{L}{h}\right)^2 \frac{1}{b}\left[0.60793\left(\frac{4}{\pi}W - V_b\right) + 1.273\left(\frac{h}{L}\right)S_b\right]\sin\frac{\pi x}{L}$ (for the beam)

Equation 8.7; $\frac{T_x}{t} = \left(\frac{L}{r}\right)^2\left(\frac{1}{t}\right)\left[\text{Constant} + F(V_b) + F(S_b)\right]\sin\frac{\pi x}{L}$ (for the shell)

We have $\frac{T_x}{t} = f_{\text{top}}$ or $\frac{T_x}{t} - f_{\text{top}} = 0$. Simplifying, we get **the first equation for compatibility of stresses by equating stresses as follows.**

$$\underbrace{\left[\text{Constant} + F(V_b) + F(S_b)\right]}_{\text{Shell}} + \left(\frac{L}{h}\right)^2\left(\frac{t}{b}\right)\underbrace{\left[0.60793\left(\frac{4}{\pi}W - V_b\right) + 1.273\left(\frac{h}{L}\right)S_b\right]}_{\text{Beam}} = 0 \quad (8.9)$$

Step 2 Write down second condition of compatibility of equality of deflection in beam and shell [Eq. (8.10)]

The deflection of beam under V_b and S_b is,

$$\delta = \frac{L^4}{bh^3E}\left[0.12319\left(\frac{4}{\pi}W - V_b\right) - 0.19351\left(\frac{h}{L}\right)S_b\right]\sin\frac{\pi x}{L} = \text{[Eq. (8.6) for beam]}$$

We get deflection of the shell Δ due to membrane forces (Table IB) and V_L, H_L, S, V_b and S_b as follows:

$$\Delta = \frac{L^4}{r^3tE}\left[\text{Constant} + F(V_b) + F(S_b)\right]\sin\frac{\pi x}{L} = \text{[Eq. (8.8) for shell]}$$

Equating the two and putting it as an equation, we get **the second equation for compatibility as,**

$$-\frac{t}{b}\left(\frac{r}{h}\right)^3\left[0.12319\left(\frac{4}{\pi}W - V_b\right) + 0.19351\frac{h}{L}S_b\right] + \left[\text{Constant} + F(V_b) + F(S_b)\right] = 0 \quad (8.10)$$

Step 3 Solve for V_b and S_b from Eq. (8.9) and Eq. (8.10).

Step 4 Having known V_b and S_b, we find stresses in beam, the area in tension. Total tension, allowable steel tension and steel area needed are calculated as shown in Example 8.2 (Step 10).

[**Note:** For **Design of edge beam**, we have to find the stresses at the bottom of the beam due to V_b and S_b. The expression for stress at the bottom of the beam will be similar to Eq. (8.9) and is as follows:

$$f_{\text{bottom}} = \left(\frac{L}{h}\right)^2\left(\frac{t}{b}\right)\left[0.60793\left(\frac{4}{\pi} - V_b\right) - 0.6366\left(\frac{h}{L}\right)S_b\right] \quad (8.11)$$

[**Note:** The second term in Eq. 8.11 is one half of the second term in Eq. (8.9) for beam part and denotes compression. This is evident from Eq. (8.3) and (8.3a).]

8.3 ANALYSIS OF SHORT SHELLS WITH HORIZONTAL EDGE BEAM

As we have already seen for short shells, the T_x forces are low. We generally provide a horizontal (cantilever) edge beam as shown in Figure 8.3. The horizontal edge beam with depth less than 1/3 the cantilever span acts as a cantilever. Whereas we took V_b and S_b in a vertical edge beam, in the cantilever edge beam, the forces taken are H_b and S_b, as shown in Figure 8.3.

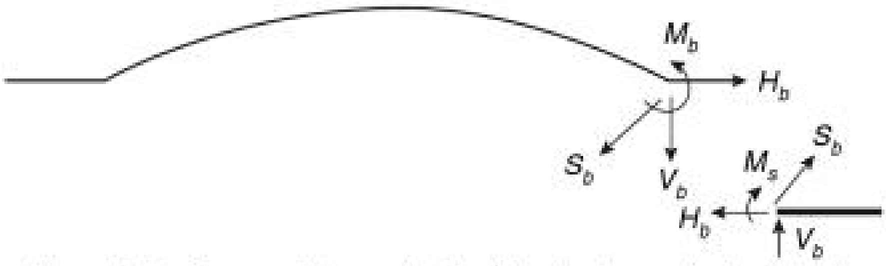

Figure 8.3 Forces acting on horizontal edge beam of short shells.

First, we carry out membrane analysis. In addition to this, we assume all the forces T_L, S, R_L and M_L act on the shell. M_L will be equal to the moment produced by the cantilever. The total V_L on the shell will be equal to the vertical load on the cantilever. Similar principle of equating T_x'/t and f and equating horizontal deflections are used in this case also.

Example 8.3 explains the boundary conditions we apply to analyze these shells.

8.4 ANALYSIS OF SOME SPECIAL TYPES OF CYLINDRICAL SHELLS

In the introduction, we listed some of the commonly used special types of shells. In the following sections, we briefly review the analysis of the remaining types of shells.

8.5 ANALYSIS OF SHELLS WITH PRESTRESSED CONCRETE EDGE BEAMS

[**Note:** In short, Section 8.5.2 gives the summary of conditions to be satisfied. The following explanations give the theory which may be omitted in the first reading.]

When the tension in the edge beam is too large, we can reduce it considerably by introducing a prestressed cable in the beam. We will assume that the cable has zero eccentricities at the ends and a downward eccentricity e at the centre of the beam so that it is parabolic and equivalent to a uniformly distributed load from below. If the magnitude of prestress is P and the middle eccentricity e, then

$$Pe = \frac{wl^2}{8} \quad \text{or} \quad w_1 = \left(\frac{8e}{l^2}\right)P$$

$$\text{Fourier loading} = w = \left(\frac{4}{\pi}\right)\left(\frac{8e}{l^2}\right)P$$

We will give prestress P of such a magnitude that the stress at the bottom fibre of the edge beam will be zero. There will be no tension.

Accordingly, there will be three unknown forces V_b, S_b and P. To solve these, we have three conditions:

1. f_{top} of beam = T_x/t of shell
2. δ of beam = Δ of shell
3. f_{botom} of beam = 0

Hence, the problem reduces to finding three equations f_{top}, δ and f_{bottom} of the beam due to the additional introduction of the prestress.

8.5.1 Influences of W, V_b, S_b and P in Prestressed Edge Beams

We will find expressions for stresses at top and bottom and also deflection due to:

(a) Effect of weight W
(b) Effect of shear V_b
(c) Effect of S_b
(d) Effect of the compressive force due to prestress $F_1(P)$
(e) Effect of the eccentricity of prestress $F_2(P)$

We will derive the values of f and δ due to each of the unknown forces with reference to Example 8.3.

(1) *Stress and deflection due to W_1 and V_b*

Let $W_1 = \dfrac{4W}{\pi}$ (weight of beam per m)

From equations derived from Example, we get V_b as the vertical reaction. In Example 8.2, the total load on the beam was $(W_1 + V_b)$. [**Note:** Factor 1.2732 as in Eq. (8.3)]

$$f_{top} = -\left(\frac{L}{bh}\right)(1.2732)\times W \text{ (or } V_b)$$

$$f_{bottom} = +\left(\frac{L}{bh}\right)(1.2732)\times W \text{ (or } V_b)$$

$$\delta = -\left(\frac{L^3}{bh^2E}\right)(0.19351)W \text{ (or } V_b)$$

(2) *Stress and deflection due to compression due to prestress P*, designated as the first effect of prestress $F_1(P)$

$$f_{top} = f_{bottom} = -\frac{(4/\pi)P}{bh}$$

Vertical deflection $\delta = 0$

(3) *Stress and deflection produced by the upward force of the parabolic prestress cable of P and central eccentricity e*, designated as the second effect of prestress $F_2(P)$

$$f_{\text{top}} = f_{\text{bottom}} \text{ and } w = \frac{4}{\pi}\left(\frac{8e}{L^2}\right)P$$

$$f_{\text{top}} = +\left(\frac{L}{bh^2}\right)(0.60793)w$$

$$\delta = -\left(\frac{L^4}{bh^3}\right)(0.12319)w$$

(4) *Stress and deflection due to shear S_b*, interaction between shell and beam

$$f_x \text{ at top } = +\left(\frac{L}{bh}\right)(1.2732)S_b.\sin\frac{\pi x}{L}$$

$$f_x \text{ at bottom } = -\left(\frac{L}{bh}\right)(0.6366)S_b.\sin\frac{\pi x}{L}$$

$$\delta = -\left(\frac{L^3}{bh^3E}\right)(0.19351)S_b.\sin\frac{\pi x}{L}$$

8.5.2 Conditions to be Satisfied

We have the following three conditions to be satisfied for equilibrium.

(1) f_x at top of beam = $F(W) + F(V_b) + F_1(P) + F(S_b) + F_2(P)$

(2) δ of beam = $F(W) + F(V_b) + F_1(P) + F(S_b) + F_2(P)$, of these $F(W)$ is known. We equate these to stress of the shell T_x/t and shell deflection ΔV. In addition, we have the condition that the stress at the bottom of the beam is zero as given below.

(3) f_x at bottom of beam = $F(W) + F(V_b) + F_1(P) + F(S_b) + F_2(P) = 0$

From these three equations, we calculate the three unknowns V_b, S_b and P. With these, we can now complete the analysis of the shell and the edge beam.

8.6 MULTIPLE BARREL SHELLS

Multiple barrel shells can be any one of the following as shown in Figure 8.4 and described below. [Reference [4] gives a good account of the various cases under this load.]

1. Multiple shells with feather edges. In this type, the inner shells have rounded edges called *feather edges* but no real edge beam. In the outer sides, there are edge beams.

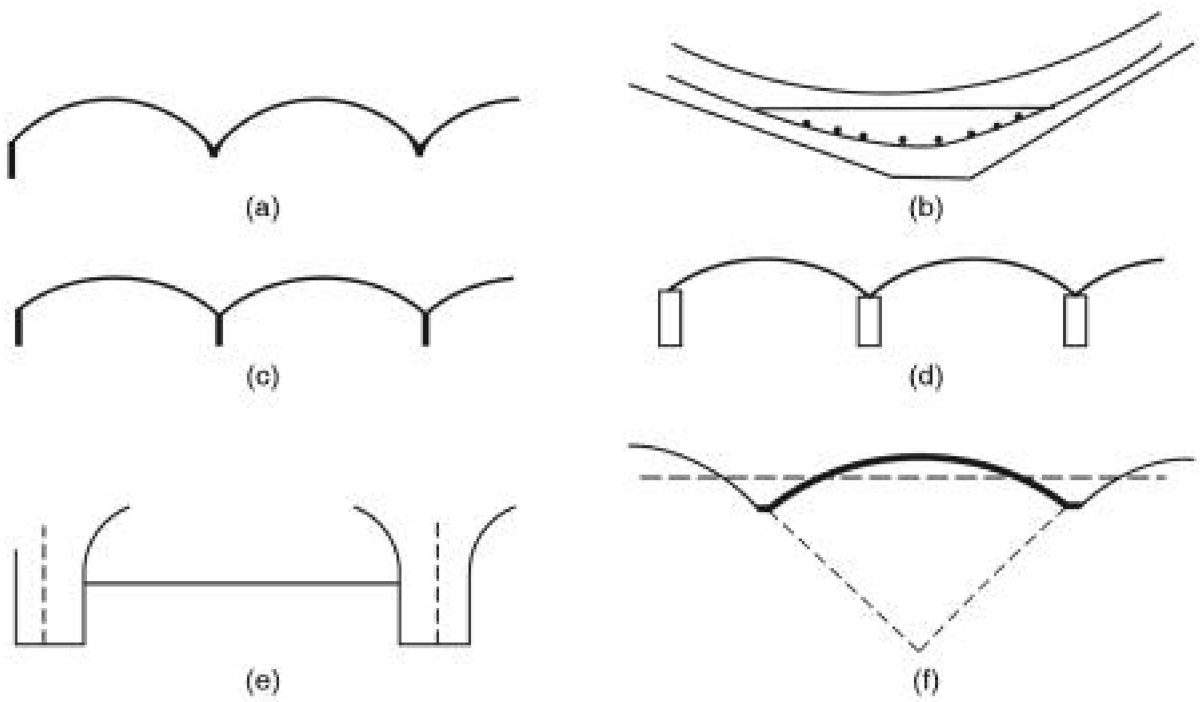

Figure 8.4 Analysis of multiple cylindrical shells: (a) Shells with feather edge beam, (b) Detailing of feather edge beam, (c) Multiple shells with small edge beams, (d) & (e) Multiple shells with regular edge beams, and (f) Beam analysis of multiple shells.

2. Multiple shells with inner shells having small edge beams which act as a tie between the shells and are incapable of bending resistance. This case is similar to case (1), but the weight of the small edge beam has to be considered in the design.
3. Multiple shells with all shells having conventional edge beams.

In general, we may say that short span multiple shells can be built with no or small internal edge beams. For long spans, it is more convenient to have regular edge beams for internal shells also.

Short descriptions of these shells are given below. For a more detailed study, consult Reference [4].

8.6.1 Multiple Barrel Shells with Feather Edge Beams—Analysis of Interior Unit

The interior junctions have no edge beams. The special detailing of the junction is sometimes referred as a *feather edge beam*. They are treated as in a symmetric problem. Several methods of solution are possible. One of them is to set up four boundary conditions producing four simultaneous equations to solve T_x, T_ϕ, S and M_ϕ. Another method is by use of formulae given in ASCE Manual No. 31.

1. $\Delta H = 0$ or the horizontal displacement of edge is zero.
2. $\theta = 0$ as no rotation can take place at the edges because of symmetry.
3. $V = 0$ as there is no member at the junction to carry the vertical load, vertical component of the forces transferred by shell to edge should be zero.
4. $S = 0$ as there is no member at the junction to carry the shear in tension at the member.

Hence, we adopt the conditions $\Delta H = 0$ and $\theta = 0$. Example 8.4 gives a brief outline of this solution.

8.6.2 Analysis of Multiple Barrel Shells with Edge Beams

As the edge beam is between two shells, we consider that only half width of the edge beam is assigned to each shell. An analysis is made as in the case of individual shell with those edge beams. The exterior half of the shell is analyzed as indicated below.

8.6.3 Analysis of Exterior Barrel of Multiple Shells

As the interior edge beam considered is only one half as stated above, the exterior half theoretically is an unsymmetric shell. A good approximation can be made by treating the exterior half of the shell as a separate single shell with full edge beam and the interior half as an interior shell with half edge beam.

8.6.4 Application of Beam Theory for Multiple Shells

We may apply the beam theory for the analysis of multiple shells with edge beams by taking one half of the edge beams on either side acting with the internal shell as shown in Figure 8.4. Reference [5] gives a worked out example using this method.

8.7 CONTINUOUS LONG SHELLS

Continuous shells are like continuous beams with more than one shell in the longitudinal direction. As discussed in ASCE Manual No. 31, the effect of continuity on shells and conventional beams is similar since it affects only the variation of longitudinal stress. Though we can make a rigorous analysis of these shells with real boundary conditions, force distribution can also be determined by proportioning the shell on the basis of the comparison of moments in a continuous beam to moment in a simply supported span [3].

8.7.1 Longitudinal Distribution of T_x

In the case of long continuous shells, we can assume it is analogous to continuous beams. The ratio of the moment in a continuous shell to the moment in a simple shell may be assumed the same as in continuous beams. We may assume that continuity does not change T_ϕ and T_ϕ forces significantly.

However, in the case of short shells because of the greater shear strain on the supporting arches, the longitudinal stresses over the supports will be somewhat greater than that indicated by the analogy of continuous beams [2].

8.8 ANALYSIS OF SINGLE SHELL WITH SMALL EDGE BEAM SUPPORTED ON A WALL ON CRAFT PAPER OR CLOSED SPACED FLEXIBLE COLUMNS

The conditions to be satisfied at the junction are:

1. Horizontal component of boundary forces is zero or $H = 0$.
2. Vertical deflection is zero or $\Delta V = 0$.
3. $M_\phi = 0$.
4. Longitudinal stress in the shell is equal to the longitudinal stress in edge member.

Reference [4] deals with this problem in detail.

SUMMARY

This chapter gives a summary of the methods to be used for the design of various types of cylindrical shells that we meet with in practice.

EXAMPLE 8.1 [Calculation of T_x/t and Δ of shell edge at $\phi = 0$ for membrane loads due to the super loads and also due to the reaction from beam V_b and S_b. This example explains calculation of T_x/t and Δ for the shell portion.]

Shell geometry: $r = 7.5$ m; $L = 25$ m; $\phi_k = 35°$; $B = 8.6$ m; ($r/L = 0.3$, long shell)

Loading: DL = 230 kg/m² – Fourier load $= \dfrac{4}{\pi} \times 230 = 293$ kg/m²

LL = 66 kg/m² – Fourier load $= \dfrac{4}{\pi} \times 66 = 84$ kg/m²

Total load = 374 kg/m² ($\approx$ 370 kg/m²)

(We may also combine LL with DL as an equivalent dead load as given in Section 8.5. Let us design for a combined dead load of 370 kg/m². This is on the safer side.)

$$\left[\frac{r}{t} = 100; \frac{r}{L} = 0.3\right]$$

Reference	*Step*	*Calculations*
		Part 1—Membrane analysis
	1.	*Find total T_x for membrane analysis*
Appendix E		$T_x = \left(\dfrac{L}{r}\right)^2 \times p \times r \times \text{Coeff.}$
Table 1B		$= \left(\dfrac{L}{r}\right)^2 \times 370 \times 7.5 \times (-0.1660) = \left(\dfrac{L}{r}\right)^2 (-460)$

Reference	*Step*	*Calculations*
		Calculate T_ϕ by membrane analysis and find V_L and V_H
Table 1B		$T_\phi = r(p \times \text{Coeff.}) = 7.5 \times 370 \times (-0.8191) = -2273$ kg
		$V_L = -2273 \sin 35 = -1303$ kg
		$H_L = -2273 \cos 35 = -1862$ kg
		Calculate S by membrane analysis
Table 1B		$S = pr\left[\frac{L}{r}\right]\text{Coeff.} = L(p \times \text{Coeff.}) = 25 \pm \times 370 \times (-0.3652) = -33$
Table 2A	2.	*Write down expression for total T_x due to all four components*
$r/L = 100$		$T_x = \left(\frac{L}{r}\right)^2 [-460 - 1303 \times 10.08 + 1862 \times 1.543 + 3344 \times 0.9676]$
$r/t = 0.3$		(Underlined values are from Table 2A.)
		This reduces to, $T_x = \left(\frac{L}{r}\right)^2 (13037)$
Eq. 8.7	3.	*Total value of T_x due to membrane analysis and V_b and S_b from beam effect (Add for V_b and S_b)*
		$T_x = \left(\frac{L}{r}\right)^2 [13037 + 10.03\, V_b + 0.9676\, S_b]$
		[**Note:** For the analysis of shells with edge beam, we find stress as T_x/t and equate it to the stress at the top layer of the beam. This result is used in Step 3 of Example 8.2.]
		Part 2—Calculation of deflection of shell
	1.	*Summary of procedure*
Table 1B		We ΔV of the membrane analysis (Table IB of Manual No. 31 and deflections due to T_x, V_L, H_L, S and V_b, S_b using Table II B of the manual as follows.
Table 2B for ΔV		ΔV = (Membrane deflection) + (That due to application of V_L, H_L and S_L)
		$= \frac{L^4}{r^3 tE}\left[pr\left(\frac{2r}{\pi L}\right)^2 + \frac{2}{\pi^4} + \left(\frac{r}{L}\right)^4 \times 0.6710\right]$
		$+ \frac{L^4}{r^3 tE}\left[V_L(37.31) + H_L(-15.65) + S(0.9583)\right]$
		(Where $V_L = -1303$; $H_L = -1862$ and $S = 3344$ from Step 1)
		$= \frac{L^4}{r^3 tE}(23679)$

Reference	*Step*	*Calculations*
Table 2B for ΔV		To the above, add effect of V_b and S_b. (Effect of beam on shell) ΔV = Total deflection = L^4/r^3tE (23679 + 37.31 V_b + 0.9583 S_b) ... Eq. (8.8) of text. (This result is used in Example 8.2.) (**Comments:** This example illustrates the computation of T_x/t and Δ of the shell. The corresponding values of the beam, f and δ can be easily written down by formulae as in Eqs. (8.9) and (8.10) as shown in the next example.)

EXAMPLE 8.2 (Analysis of the long shell in Ex. 8.1 with conventional bottom edge beams)

Shell data as in Ex. 8.1 – L = 25 m; ϕ_k = 35°; r = 7.5 m; t = 75 mm (same as in Ex. 8.1.)

Edge beam—(1.5 m × 0.25 m) in size (Depth = $L/16$) (approx.)

Beam Wt. = 0.25 × 1.5 × 2400 = 900 kg/m; Sinusoidal wt. = $4/\pi$ × 900 = 1146 kg/m

Total load of the shell with water proofing = 370 kg/m^2

$$\sin \phi_k = 0.5736;\ \cos \phi_k = 0.8192$$

Chord = $2r \sin \phi_k$ = 8.6 m; rise = 1.36 m = 1/5.88 of chord width 1

$$\frac{t}{L} = \frac{7.5}{25} = 0.3; \frac{r}{h} = \frac{7.5}{1.5} = 5; \left(\frac{r}{h}\right)^2 = 25; \left(\frac{r}{h}\right)^3 = 125$$

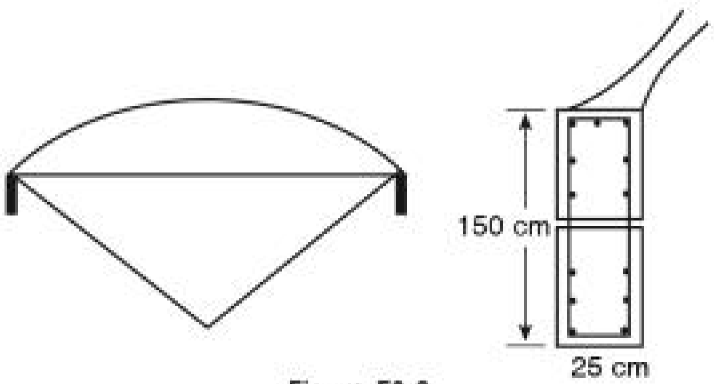

Figure E8.2

Reference	*Step*	*Calculations*
	1.	*Write down total stress on shell*
Ex. 8.1,		From Ex. 8.1,
Eq. (8.7)		$\dfrac{T_x}{t} = \left(\dfrac{L}{r}\right)^2 \left(\dfrac{1}{t}\right)[13037 + 10.03V_b + 0.9676S_b]$
	2.	*Equating T_x/t to stress at top of beam (f_{top}) and reducing it to an equation by using Eq. (8.9) [For f and δ of beam, use equations directly]* (Let 13037 + 10.03 V_b + 0.9676 S_b be called part A)

Reference	*Step*	*Calculations*
Eq. (8.9)		$A+\left[\left(\frac{r}{h}\right)^2\left(\frac{t}{b}\right)\left(0.60793(1146-V_b)+1.273\left(\frac{h}{L}\right)S_b\right)\right]=0$ $A + (5220 - 4.559V_b + 0.5729S_b) = 0$ Or $5.47V_b + 1.54S_b = -18267$ $3.55V_b + S_b = -11861$ Eq. (1)
	3.	*Calculate vertical deflection of shell* (1) Due to membrane forces, Table IA and IB (2) Due to V_L, V_H and S, Table 2B (3) Due to V_b and S_b on the shell [**Note:** We apply $-V_b$ and $-S_b$ on the beam.]
Ex. 8.1		$\frac{L^4}{r^3tE}[\text{Constant} + F(V_b) + F(S_b)]$ Take value from Example 8.1. $= \frac{L^4}{r^3tE}[23679 + 37.31V_b + 0.9583S_b]$ Equating the deflection of the shell to the deflection of the beam, we get from Eq. (8.10), ($20739 + 37.31V_b + 0.9583S_b$ – let this be called part B)
Eq. (8.10)	4.	*Equate deflections of shell and beam* $B+\left[-\frac{t}{b}\left(\frac{r}{h}\right)^3\left[0.12319(1146-V_b)+0.1935\left(\frac{h}{L}\right)S_b\right]\right]$ $B - 5294 + 4.62V_b - 0.4354S_b = 0$ Or $41.93V_b + 0.5229S_b = -17.445$ Or $80.2V_b + S_b = -33.362$ Eq. (2)
	5.	*Solve for V_b and S_b* $80.20V_b + S_b = -33362$ (2) $3.55V_b + S_b = -11861$ (1) $76.65V_b = -21501$ Hence, $V_b = -280.5$ kg and $S_b = -10865$ kg [**Important note:** The sign of V_b is –ve. Hence the load on the beam is $(W + S_b)$. Some books hence start with S_b acting in the opposite direction as we have assumed in this chapter in Figure 8.2.]

Reference	*Step*	*Calculations*
	6.	*Calculate stress at top and bottom of beam due to total weight*
		Total vertical load = Wt. of beam + V_b
		= 1146 + 280 = 1426 kg/m
Eq. (8.2)		$f = \mp V_b (0.60793)\left(\frac{L}{h}\right)^2 \times \frac{1}{b} \sin\frac{\pi x}{L}$
Eq. (8.11)		$f = \mp 1427 (0.60793)\left(\frac{25}{1.5}\right)^2 \left(\frac{1}{0.25}\right)$
		$= \mp 963906$ kg/m^2 = 96.4 kg/cm^2 (Comp. at top of beam)
	7.	*Calculate stresses in beam due to S_b*
Eq. (8.3)		Stress at top edge = $(L/bh)(1.2732)S_b$ (tension)
Eq. (8.3a)		[Stress at bottom is one half of the above [Eq. (8.9b)]
		$= -\frac{L}{bh}(0.6366)S_b$ (Compression)]
		Tension at top = $\left(\frac{25}{0.25 \times 1.5}\right)(1.2732)(10865)$
		= 922221 kg/m^2 = 92 kg/cm^2
		At bottom = 46 kg/cm^2 (Compression at bottom)
	8.	*Resultant stresses in beam due to V_b + S_L*
Step 7 + 8		Compression at top of beam = 96.4 − 92 = 4.2 kg/cm^2
		Tension at bottom of beam = 96.4 − 46 = 50.4 kg/cm^2
	9.	*Design of steel reinforcement in edge beam*
		Depth of beam in tension with above stress distribution in 1.5 m depth of beam
		$= \frac{150 \times 50.4}{54.6} = 139.4$ cm
		Total tension = $\frac{1}{2} \times 139.4 \times 50.4 \times 25$ = 87,682.6 kg
		We can adopt the simple rule that total steel area = (Total tension ÷ Allowable stress) even though it is not strictly correct. (see Sec. 9.4.)
		Assuming 10 cm cover and assuming stress at extreme fibre of beam is 2400, stress at the level of the first layer of steel with 10 cm cover is,
		Allowable stress = $f_s = \frac{2400 \times 129.4}{139.4} \cong 2225$ kg/cm^2
		$A_s = \frac{87682}{2225}$ = 39.4 cm^2. This steel is provided in the tension zone.

Reference	*Step*	*Calculations*
		Provide 5 Nos. of 25 ϕ (24.54 cm^2) + 4 Nos. 22 mm (15.21 cm^2) = 39.72 cm^2 at bottom of the beam. In regions where there are no steel, provide 10 mm nominal at maximum spacing.
		Provide also 16 mm nominal stirrups.
	10.	*Design the shear steel in beam*
		$V_b = 280 \sin \pi x/L$ (interactin with shell)
		Max. shear due to V_b acting over one half the length of beam,
		Total load
Data		Shear self weight = $L/2 \times W = 12.5 \times 900 = 11250$ kg
V_b is –ve		Total shear in beam = $W + V = 13480 = V_1$
		Spacing for a stirrup of area A_s with two legged stirrups,
		$s = \dfrac{(2A_s)\times 0.87 f_y}{(V_1/d)}$ assuming all shear is taken by steel.
		Choosing 16 mm two legged stirrups assuming all shear is carried by steel using MS and *effective depth* 125 cm,
		$s = \dfrac{2\times 2.04\times 0.87\times 1400}{13480/125} = 46$ cm spacing

EXAMPLE 8.3 [Design of a short shell with horizontal edge beam (member) used for hangars and auditoriums]

[**Note:** This is a worked out example in Manual No. 31. Only method of analysis is described below.]

Figure Example 8.4 shows a *combination shell* consisting of two different curvatures with horizontal edge beams. The central shell (called upper shell) has a large radius of curvature and the shells on both sides (called the lower shell) have lower radius, the change in curvature taking place at $\phi = 35°$. Indicate how to design the shell roof. [Refer Example 7 of ASCE Manual No. 31.]

[**Note:** Part (1) briefly indicates the design of the shell part. Part (II) illustrates the design of the cantilever edge beam.]

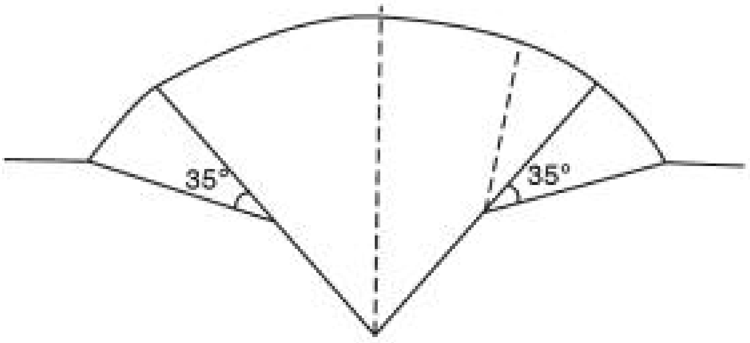

Figure E8.3 Combined shell with horizontal edge beam for a hanger.

Reference	*Step*	*Calculations*
		Part 1—Analysis of the composite shell
	1.	*Determine the force and displacement components at the boundaries of the shell due to surface loads by membrane analysis*
		[**Note:** As we are dealing with a short shell, the membrane forces are T_x, T_ϕ, S from Table 1B. Table 3A gives the effect of line load T_L, R_L, S_L and M_L and deformations u (longitudinal direction), v (tangential direction), w (radial direction) and θ (rotation). We have to make forces and deformations compatible at: (a) Junction of upper shell and lower shell (b) Junction of lower shell and cantilever edge beam]
	2.	*Introduce line loads*
		The values of the forces of the two shells obtained by membrane analysis at the junctions $\phi = 35°$ will be different.
		Introduce T_L, S_L, R_L and M_L on the upper edge of the lower shell and equal and opposite forces on the lower edge of the upper shells as unknowns. These *four unknowns* are to be determined by the following conditions: (1) Longitudinal forces in both shells are equal. (2) Tangential displacements are the same. (3) Radial displacements are the same. (4) Rotations are the same.
	3.	*Find effects of line loads on shell*
		The effects of these line loads at various distances from the junctions are obtained by use of coefficients in Tables 3A and 3B and the forces in the upper shell can be determined.
		Part 2—Analysis of cantilever edge beam in short shells
		[**Note:** This problem is similar to that of vertical edge beam and we will examine it in more detail.]
		Part 1 analysis was for the shell. In addition, we have to find out the effect of the cantilever projection on the shell. We use Tables 3A and 3B of Manual No. 31.
		Accordingly, all the four line loads T_L, S_L, R_L and M_L corresponding to all loads in Table 3B of Manual No. 31 (Appendix E) are to be applied as correction loads on the edge of the lower shell (junction of shell and cantilever).
	1.	*We first consider the shell and the conditions of the shell due to loads and the action of cantilever beam*
		M_L and R_L are taken as unknowns, we consider the following:
		(1) The moment M_b in the shell. It should be equal to the moment applied by the cantilever slab (M_L).

Reference	*Step*	*Calculations*
		(2) By statics, the vertical component of the correction line loads must be equal to the total load from cantilever. Hence, T_L and R_L are functions of vertical load.
		Hence, $T_L[\sin(\phi_k - \phi)] - R_L[\cos(\phi_k - \phi)]$ = Cantilever vertical load
		Hence, T_L can be found in terms of R_L.
		We find T_x produced by:
		(a) Application of line loads from membrane analysis (b) Due to M_L due to weight of cantilever (c) Due to S_L (which is unknown) (d) Due to R_L (which is unknown) (e) Due to T_L which can be expressed in terms of R_L (as above)
		The stress in the shell T_x/t can be found,
		T_x/t = Deformation along x-axis
Table 3B		(3) We can also calculate ΔH of shell = $v\cos(\phi_k - \phi) - w\sin(\phi_k - \phi)$.
		where v and w are tangential and radial displacements of shell due to all the loads.
		From these values, we write the expressions for ΔH and T_x/t of the shell at the edge.
	2.	*Next we consider the horizontal beam*
		(4) The forces acting *on the beam from the shell* are only H_b (horizontal load) and shear S_b.
		Equating horizontal forces
Fig. 8.3		$H_b = T_L\cos(\phi_k - \phi) + R_L\cos(\phi_k - \phi)$, which will be a function of R_L
		Also, $S_b = -S_L$ from shell
		From these, we can calculate the following:
		δ_h = Horizontal deflection of beam in terms of R_L and S_L
		f = Stress in the fibre of beam near shell in terms of R_L and S_L
	3.	*Compatibility conditions*
		The two compatibility conditions chosen in terms of R_L and S_L are:
		(a) T_x/t of shell must be = f of cantilever beam. This is in terms of R_L and S_L.
		(b) ΔH, horizontal displacement of the shell and that of the beam δ_H are the same.
		Having determined the value of the forces, S_L and R_L from the above compatibility conditions, the shell and cantilever can be separately designed.
		The horizontal edge beam is assumed to carry its own weight as a cantilever. The shell has to be analyzed for the forces introduced by the edge beam on the shell.
		(Refer ASCE Manual No. 31 for complete analysis.)

EXAMPLE 8.4 [Analysis of the interior shell of multiple cylindrical shell with feather edge beam or no edge beam]

Geometry of shells—L = 25 m; r = 7.5 m; $\phi_K = 35°$; t = 75 mm; r/L = 0.3 (long shell)

Loading—Assume equivalent dead load = 300 kg/m^2; $(4/\pi \times 300)$ = 380 kg/m^2

(Refer Figure 8.4)

Reference	*Step*	*Calculations*
	1.	*Calculate membrane stresses*
		Use Table IB (b) of Manual No. 31 given in Appendix E or formula.
	2.	*State boundary conditions*
		Conditions to be satisfied with reference to Section 8.6.1, $\Delta H = 0$ and $\theta = 0$. Hence introduce H_L and M_L along the edges as the unknowns. Vertical line load V_L and shear line load S are present as in the case of simply supported cylindrical shells.
	3.	*Condition* $\Delta H = 0$
Table 2B		Find ΔH using Table 1B and Table 2B for the values of T_x, S_L and S as got from membrane analysis plus that due to the unknown H_L and M_L. Equate it to zero.
		$\Delta H = L^4/tEr^3$ (due to T_x, S_L, S and H_L and M_L)
		i.e. it reduces to $20{,}495 - 7.89\,H_L - 54.5(M_L/r) = 0$
	4.	*Condition rotation* $\theta = 0$
		$\theta = r^2/EI$ [due to T_x, S_L, S, H_L and M_L.]
		It reduces to $142.54 - 0.05606\,H_L - 0.4965(M_L/r) = 0$
	5.	*Find* H_L and M_L
		Solving the two equations, we get
		H_L = 2787 kg/m and (M_L/r) = –27.45 kg/m
	6.	*Analyze the shell for these forces also*
		The sum of the membrane and edge force analysis gives the final forces on the shell.

EXAMPLE 8.5 (Analysis of multiple barrel with regular edge beam)

The most convenient method to analyze these shells is to consider them as single shells with edge beams.

For the middle shells in the multiple shell group, we consider only one half edge beam on either side to act with the interior shells.

The exterior half of the exterior shell is designed as part of a single shell with the same exterior edge beam. Though this procedure is theoretically not correct, it gives safe values in practice.

REVIEW QUESTIONS

1. What types of circular cylindrical shells are commonly used?
2. What types of edge beams are used for long shells and short shells? Give reasons for such shapes.
3. Describe briefly the method of analysis of long shells with edge beams.
4. When do you use prestress in edge beams? What modification in the analysis you make for this effect?
5. What are multiple shells and continuous shells?
6. What types of multiple shells are used in practice?
7. How do you analyze multiple long shells with regular edge beams?

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9

DETAILING OF STEEL IN CYLINDRICAL SHELLS

9.1 INTRODUCTION

For moderately sized cylindrical shells, most of the areas are in compression and the necessary reinforcement required will be found to be only nominal. But it is very important that the shell is reinforced properly for the various types of forces in it. In all cases at least the minimum amount of steel specified in codes as given in Section 9.3 should be provided in all parts of the shell.

9.2 GENERAL ARRANGEMENT OF STEEL

In early days of development of the theory of cylindrical shell design, it was the practice to reinforce the shell along the lines of the principal tension. This required placement of steel is a curved pattern. The modern practice is to provide straight bars longitudinally for T_x forces, transversely for T_ϕ and M_ϕ forces and diagonally for S forces. *For longitudinal and transverse minimum steel, we may use steel fabrics.* We have already studied in Chapter 5 about the distribution of the principal forces T_x, T_ϕ, M_ϕ and S in these shells. The following are the sets of steel used. They are detailed as shown in Figures 9.1 and 9.2 for the forces.

1. One set of reinforcement for the force T_x along the longitudinal direction in all places and especially in the areas of tension.
2. One set of steel for forces T_ϕ and M_ϕ in the transverse direction. This steel is provided on the side where M_ϕ produces tension.
3. One set of steel for the shear force S, as diagonal steel as shown in Figure 9.1, is placed in a manner similar to diagonal steel in beams from the tension side to the compression side diagonally.

4. We must also detail the junctions between the shell and the transverse and the shell and the edge beam properly. We must remember that the transverse at the support of the shell and the edge beams are all integral parts of the shell. The steel should extend from one part to the other part of the shell.
(In some detailing practice, steel fabrics are placed at the top and bottom for minimal steel for T_x, T_ϕ and M_ϕ forces. Further extra steel necessary is provided as steel rods.)

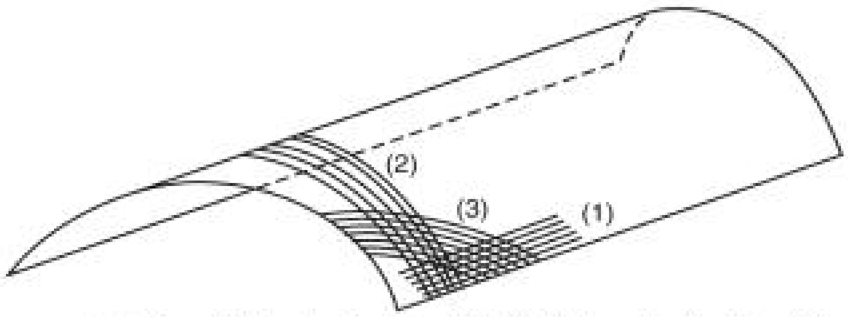


Figure 9.1 Main reinforcements in the shell part of a barrel shell: (1) Longitudinal steel for T_x forces, (2) Transverse (circumferential) steel for T_ϕ and M_ϕ forces, and (3) Shear steel for S forces.

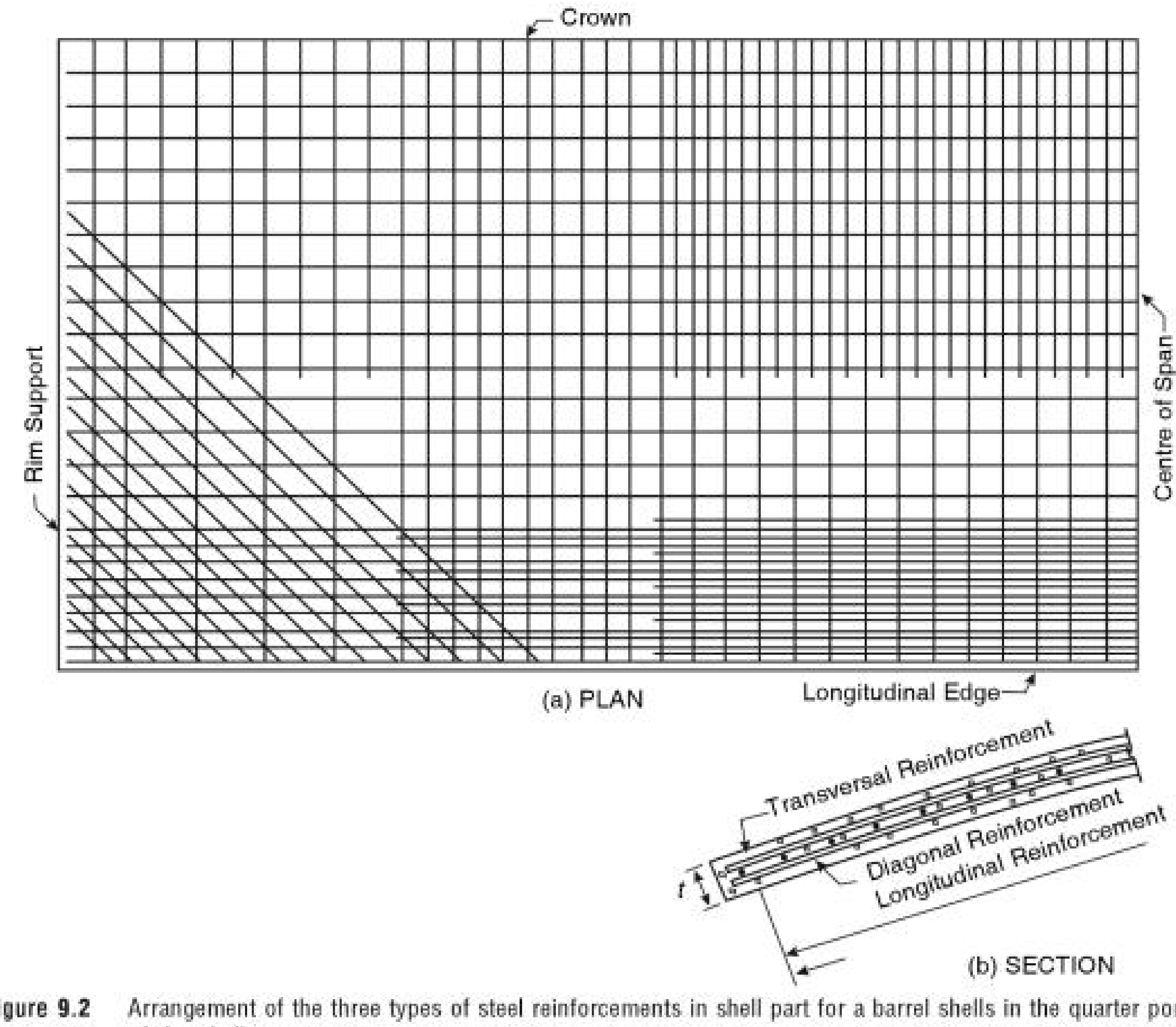


Figure 9.2 Arrangement of the three types of steel reinforcements in shell part for a barrel shells in the quarter portion of the shell.

9.3 MINIMUM AMOUNT OF STEEL RECOMMENDED IN SHELLS

The minimum amounts of steel specified in cylindrical shells are as follows [1]:

1. In tensile zones, a minimum area of 0.35% in each two directions longitudinally and transversely for crack control.
2. In compression zone, ACI recommends a minimum steel that is usually provided in slabs, namely, 0.15 to 0.18 percent of steel.
3. The spacing of reinforcement should not be more than five times the thickness of the shell or 450 mm whichever is less as in placing of steel in slabs.

As already pointed out, for minimum steel we can provide steel fabrics (such as 150 × 150 mm 8/8 mm up to 150 × 150 mm 12/12 mm) on top and bottom faces. Additional steel as calculated can then be placed between them as steel rods.

IS 2210–1988 recommends that steel should have a minimum diameter of 5 mm and the maximum diameter should not exceed 10 mm in areas of shell 40 to 50 mm in thickness and 12 mm for areas of shells where the thickness is 55 to 65 mm thick. A maximum diameter of 12 mm is allowed in thickness larger than 65 mm. (Generally, we use shell thickness of 75 mm to 100 mm in practice.)

It is also specified that:

(a) The area of unreinforced part in any part of the shell should not exceed 15 times the square of the thickness of the shell. (The spacing of steel should not be more than 5 times the thickness as already stated.)

(b) The total depth occupied by the steel in the direction of the thickness in the unthickened part of the shell shall not exceed 3 times the maximum permissible diameter of the steel rod used.

(c) Even though M_x is ignored in the analysis of cylindrical shells, nominal reinforcements should be provided at the ends of the shell for these forces.

(d) The minimum cover should be 13 mm or size of reinforcement.

It is also to be noted that we analyze the shell assuming it is homogeneous. Theoretically, the steel stress should be times the concrete stress. However, for all practical purposes, in shells, as in other structures, we assume that reinforcement is stressed to an allowable design value only using elastic design.

A safe value to be adopted is that we use for liquid retaining structures as follows:

1. Direct tension 115 N/mm^2 ($0.44f_y$) for Fe 250 plain bars and 150 N/mm^2 ($0.36f_y$) for Fe 415 deformed bars. However, we can also safely use a value of 230 N/mm^2 for Fe 415 steel, the value used for ordinary structures.

For the limit state method, we use a load factor of 1.5 for the calculation of forces and an allowable stress of 0.87 f_y (For Fe 415 steel, it will be 0.87 × 415 = 360 N/mm^2. (Its equivalent for working load will be 360/1.5 = 240 N/mm^2 in tension, which is rather high.)

9.4 LONGITUDINAL STEEL FOR T_x FORCES AND FOR EDGE BEAMS

T_x is the force in the shell proper in the horizontal direction. For these forces, steel is placed in the longitudinal direction. If a long cylindrical shell is assumed to act as a beam, there will be tension below the neutral axis. The edge beam is also in tension. There are several methods of proportioning the steel in the edge beams [2].

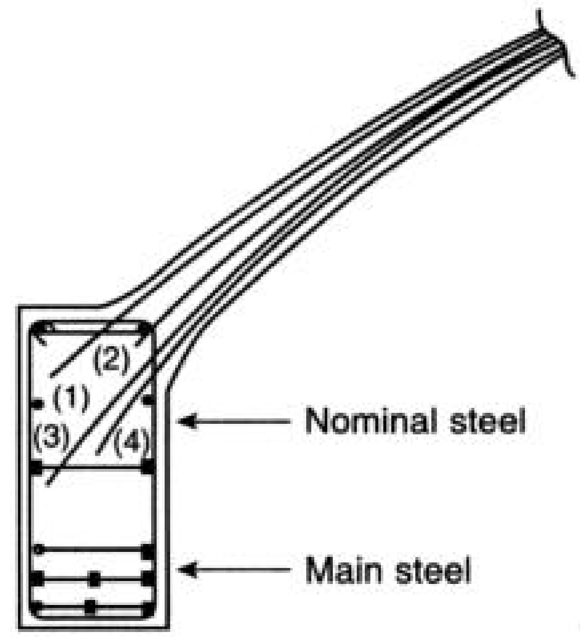


Figure 9.3 Detailing of edge beam of long shells with steel fabrics: (1) Top fabric, (2) Transverse steel, (3) Diagonal steel, and (4) Bottom fabric.

The first method for placing *steel in edge beams* of long shells is to place the steel *as in ordinary beams* as far from the neutral axis as possible, the area of the steel being computed on maximum allowable stress. This is shown in Figure 9.3. This will give minimum steel (see Example 8.2, Step 10).

The second method is that recommended by ASCE Manual No. 31. As the tension stresses vary from the maximum value at the edge of the shell to zero at the level of the N.A., the tensile steel is distributed all over the tension zone (unlike that in a beam). The allowable tension also should vary with the distance from the N.A. The allowable stress allowed for the most distant part is the maximum allowable value. For the other regions, the allowable stress is reduced in proportion to its distance from the neutral axis. For this case, we may use a higher allowable value for maximum stress.

In the third method, the area of steel required is given by the total T_x value of the region divided by a constant—maximum allowable stress allowed at the centroid of the tension area. The required steel is placed with the centre of gravity of steel area at or below the centre of gravity of the tensile force in the shell.

As a rough rule, considering edge beam reinforcement along the span, every fourth bar may be terminated at $L/4$ from the support. Only 50% of the steel is carried beyond $L/6$ of the span from support. In any case the minimum specified steel should be maintained at all sections.

According to the second method, as illustrated by Example 9.1, we take the allowable stresses as follows. For a shell with depth of N.A., say, 70 mm, we divide it into, say, 3 areas.

Region 70 cm to 40 cm to N.A. (neutral axis)—T_x/allowable stress (f_s)

Region 40 cm to 20 cm—Allowable stress = $f_s \times 40/70$

Region 20 cm to N.A. (neutral axis)—Allowable stress = $f_s \times 20/70$

9.5 TRANSVERSE STEEL FOR T_ϕ AND M_ϕ FORCES

The general distribution of T_ϕ and M_ϕ at midspan is generally as described in Chapter 5, but the actual distribution depends on r/L ratios.

Transverse steel is computed for moment M_ϕ and direct force T_ϕ. The procedure to be used is similar to that used for reinforced concrete beams subjected to combined bending and compression and combined bending and tension.

If M_ϕ produces tension at the top, then steel has to be placed near the top of the shell. For combined tension due to T_ϕ and moment M, we have to design the section for a moment equal to $M + \Delta M$ as explained in standard books on Reinforced Concrete.

As both T_ϕ and M_ϕ vary along the length, they are assumed to vary as $\sin \pi x/L$, curtailment of steel towards the support is possible. Usually, curtailment is made at $L/6$ and $2L/6$ from the centre of the shell. Generally, we need to provide only 50% of the reinforcement provided at the centre beyond $2L/6$ from the support.

9.6 STEEL FOR SHEAR S

Generally, shear is maximum near the supports. The layout of diagonal steel is as shown in Figure 9.1. It can be designed by several methods.

The exact method is to provide it as the basis of principal stresses calculated for various points. Such a method assumes that the steel is placed along the stress trajectories which are not convenient for field placement of steel.

The principal stresses are calculated from the standard formula at different points at ends $L/4$, $3L/8$ and $L/2$ as follows:

$$N_1 = \frac{T_x + T_\phi}{2} + \left(\frac{1}{2}\right)\sqrt{4S^2 + (T_x - T_\phi)^2} \tag{9.1a}$$

$$N_2 = \frac{T_x + T_\phi}{2} - \left(\frac{1}{2}\right)\sqrt{4S^2 + (T_x - T_\phi)^2} \tag{9.1b}$$

If we neglect T_x and T_ϕ, then the principal stresses are proportional to S only.

In the exact method, we have to calculate the principal stresses and stress trajectories for placement of tension steel. In the approximate method, we place it at 45° (diagonally between support and the edge beam). This approximate method is to design it by the conventional method which we use in R.C. (reinforced concrete) beams [2]. Accordingly, steel is provided only to withstand shear over and above a fixed value of allowable shear in concrete equal to V. If shear is S, then the steel required for shear will be that necessary for the net shear to be taken by steel.

$$A_s = (S - v_c \times t \times 1) \div f_s \tag{9.2}$$

where v_c = Allowable shear in N/mm^2 (about 0.4 N/mm^2 for 0.3% steel)

t = Thickness in mm

f_s = Allowable stress in steel

In practice, we may assume zero shear to be taken by concrete and provide steel for S, the full shear. It is desirable to compute shear at support and other critical points and provide steel at 45° to the direction of its length of the shell as shown in Figure 9.1.

We may divide the diagonal steel placement into regions as shown in Figure 9.2 on the following basis [2]. It can be seen that for a particular layout of the shell, a diagonal steel starting at $\phi = 10°$ at $x = 0$, may be passing over $\phi = 0°$ at $x = L/8$. Similarly, the diagonal at $\phi = 20°$ at $x = 0$, may pass through $\phi = 10°$ at $x = L/8$, and steel at $\phi = 30°$ at $x = 0$ may pass through $\phi = 20°$ at $L/8$. Hence, we may place the diagonal steel by first dividing the area into regions along the diagonal and prescribing the steel for each region (see Example 9.1). [Reference 3 explains design of cylindrical shells in simple terms.]

9.7 DETAILING OF JUNCTION BETWEEN SHELL AND TRANSVERSE AND EDGE BEAM

The transverse beam is an integral part of the shell as explained in Chapter 10 when dealing with its design. The detailing of the reinforcement in the edge beam and the transverse beam is shown in Figure 9.3. Usually, the shell is thickened at the boundaries. In singly curved shells, this thickness is carried out over a distance of 0.38 to $0.76\sqrt{rt}$, where r is the radius and t the thickness of the shell.

9.8 CONSUMPTION OF STEEL

It is claimed that the consumption of steel in cylindrical shells per square metre of plan area can be estimated by the following formula [4]:

$$\text{Quantity of steel in kg/m}^2 \text{ of plan area} = \left[\frac{L(L+B)}{20h} + 6\right] \tag{9.3}$$

where L = Span

B = Chord length

h = Rise above springing to the crown

SUMMARY

It is very important that we should detail the reinforcements in cylindrical shells according to the conventional system. Though in earlier days shell reinforcement were placed along the lines of principal stresses due to the combined forces, nowadays we provide steel separately for the major forces T_x, T_ϕ, S and M_ϕ. It is also necessary that all parts should have at least the minimum steel at specified spacings given in I.S. (Indian Standard) codes.

EXAMPLE 9.1 Design of Reinforcement in Long Shells

The final data from the analysis of a shell without edge beam (Example 5.2) gave the following results (see Example 5.2) at various ϕ values for T_x, T_ϕ and M_ϕ at $x = L/2$ and S at $x = 0$ or L:

ϕ	0	10	20	30	40	
T_x	+63579	–2230	–14117	–6167	–869	(kg/m)
T_ϕ	0	–1125	–2494	–2981	–2946	(kg/m)
M_ϕ	0	+36	–251	–423	–468	(m.kg/m)
S	0	–6752	–3774	–1303	0	(kg/m)

Indicate how to design reinforcements for these forces assuming f_s = 2300 kg/cm^2.

Reference	*Step*	*Calculations*
		Part I—Design for T_x forces in the shell
	1.	*Plot T_x forces with depth. Find depth of neutral axis. Divide depth into 3 ones. Find allowable stress for elastic design.*
		Total rise of the shell = 175 cm. N.A. is 70 cm from the bottom. Divide depth below N.A. into the following regions:
		(a) Region 1—From bottom to 20 cm above bottom
		Allowable stress = 2300 kg/cm^2
		(b) Region 2—From 20 cm to 40 cm above bottom
		Allowable stress = $\dfrac{2300\times(70-20)}{70}$ = 1643 kg/cm^2
		(c) Region 3—From 40 cm above bottom to NA
		Allowable stress = $\dfrac{2300\times(70-40)}{70}$ = 986 kg/cm^2
	2.	*Find the steel area in* cm^2/meter *length for each region*
		Steel area/per metre length = $\dfrac{T_x \text{ force in region}}{\text{Allowable stress}}$
		(a) Region 1—$A_{st} = \dfrac{63.579}{2300}$ = 27.6 cm^2/per metre length
		Provide 16 mm bars @ 70 mm spacing (28.7 cm^2).
		(b) Region 2—From plot value of T_x = 33,000 kg/metre
		$A_s = \dfrac{33,000}{1643}$ = 20 cm^2
		Similarly, for Region 3 also.
		(Nominal steel per meter length – thickness of shell 7.5 cm)
		A_s (nominal) = $\dfrac{0.15\times100\times7.5}{100}$ = 1.13 cm^2/m
		Spacing to be less than 5 × thickness or 450 mm.
		6 mm @ 25 cm gives 1.13 cm^2/m.

Reference	*Step*	*Calculations*
	3.	*Curtailment of bars*
Sec. 9.4		We can cut off every fourth bar 1/4th the span from the support and only 50% of steel is provided for region between 1/6th the span and the end of the span.
		Part II—Design for T_ϕ and M_ϕ
		We will provide steel for T_ϕ and M_ϕ separately.
		T_ϕ is mostly compression. Use 10 mm rods.
		Max. force at $\phi = 40°$ is 2946 kg/m.
Sec. 9.5		Stress $= \dfrac{2946}{100 \times 7.5} = 3.9$ kg/cm^2 only
		Design for max. $M_\phi = -468 \times 100$ cm.kg/m
		Effective depth = 7.5 – Cover (2 cm) – 0.5 = 5 cm
		L.A. (Lever arm) = 0.87 × 5 = 4.35 cm
		$A_{st} = \dfrac{46800}{2300 \times (0.87 \times 4.35)} = 5.37$ cm^2/m
		Provide 10 mm @ 15 cm (5.23 cm^2/m)
		(The compression helps M_ϕ.)
		Alternatively, we design for combined bending and compression.
		Part III—Design for Shear (*S*)
Sec. 9.6		Theoretically, the diagonals should be designed for principal tension. As an approximation, let shear = tension.
		As the shear steel is placed at 45° (*from the tension side as the diagonal steel for shear in a beam*), the steel for shear at $f = 15°$ at the end ($x = 0$) will pass through $\phi = 0°$ at $x = L/8$.
		Divide shear at support into regions (a) $\phi = 10°$ to 20° (First Region) (b) 20° to 25° (2nd Region), and (c) 25° to 40° (3rd Region)
		(a) *Steel for first region* – $\phi = 10°$ (at support)
		We can calculate principal tension and provide steel. But to make it simple, we will assume shear = tension.
		S @ 10° = 6752
		$A_{st} = \dfrac{6752}{2300} = 2.9$ cm^2/m
		10 mm @ 25 cm gives 3.14 cm^2.
		(*b*) *Steel at* $\phi = 20°$
		S @ 20° = 3774

Reference	*Step*	*Calculations*
		Steel for tension = $\dfrac{3774}{100 \times 7.5}$ = 5 kg/cm^2 As the tension is low, we can adopt nominal steel. [**Note: Detailing of Edge Beams**—Section 9.3 of this chapter explains detailing of edge beams. Example 8.2, Step 10 gives an example for the design of edge beams. In this procedure, the *total tension* in the beam is calculated and the beam is detailed as an ordinary beam. See also Figure Ex.8.2 and Figure 9.3.]

REVIEW QUESTIONS

1. Where do you use steel fabrics as reinforcement for cylindrical shells?
2. Indicate how you provide for shear. Give details as to how the shear steel is terminated at the ends in the edge beam and the transverse.
3. How do you detail the junction between a long shell and (a) its edge beam and (b) its transverse? Described in Chapter 10.

REFERENCES

[1] Design of Cylindrical Shell Roofs, Manual No. 31, American Society of Civil Engineers, 1958.

[2] Billington, D.P., *Thin Shell Concrete Structures*, McGraw Hill, New York, 1965.

[3] Design of Barrel Shell Roofs, Concrete Information Series, Portland Cement Association, Chicago.

[4] Ramaswamy, G.S., *Design and Construction of Concrete Shell Roofs*, McGraw Hill, New York, 1968.

10

DESIGN OF TRANSVERSE STIFFENERS OF CYLINDRICAL SHELLS

10.1 INTRODUCTION

Cylindrical shells are fixed at their ends to transverse stiffeners (also called diaphragms), which can be a beam or frame as shown in Figure 10.1. It is important to realize that these supports at the end of the shell are not like the support of a beam but are parts of the shell and act with the shell. The major forces acting on this *rigid member* are its own weight and, in addition, the direct force exerted by the end of the shell it supports.

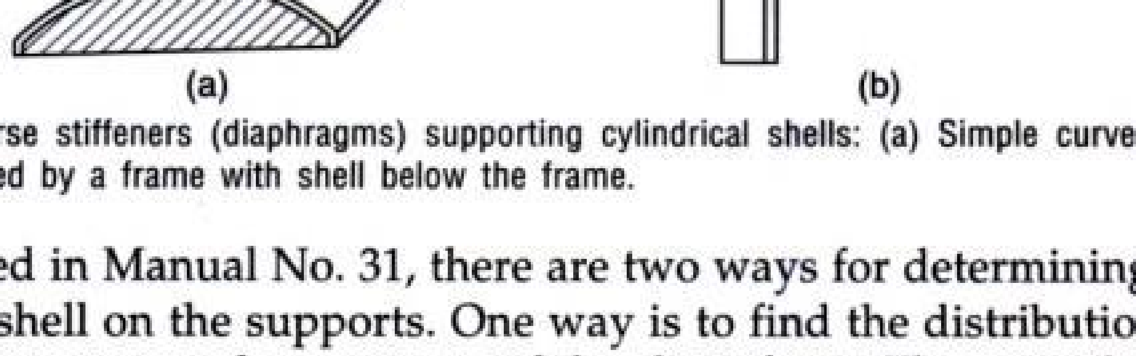


Figure 10.1 Transverse stiffeners (diaphragms) supporting cylindrical shells: (a) Simple curved beam, and (b) Shell supported by a frame with shell below the frame.

As explained in Manual No. 31, there are two ways for determining the effect of the shears from the shell on the supports. One way is to find the distribution of shear on the supports and then sum up the moments of the shear force. The second method, which is easier, is to calculate the effect of the shear from the surface loads that cause the shear. This method is recommended for design purposes in Manual No. 31. The effect of the stiffener on the shell is not considered in our analysis.

We have already seen in Chapters 6 and 7 that the long shell acts as a beam and the short shell as an arch. The end transverses will also have to be planned differently. When we plan a short shell layout for an auditorium or aircraft hangar, the shell will have to be supported on arches. The design of these arches should enable the shears to be carried to the ground. In this chapter, we discuss the stiffeners used for long shells only. These can be stiff beams of different shapes supported freely, or columns or frames connected rigidly to columns. When we plan multiple shells for a column free space, we can combine the end stiffeners of the various shells into a beam or frame which needs to be supported only at the ends.

10.2 DESIGN OF TRANSVERSE STIFFENERS (DIAPHRAGMS) OF LONG SHELLS

As shown in Figure 10.1, the supports of long shells act with the shell like a *T*-beam. When shells are on both sides of the transverse, the effective width of the *T*-section is to be taken as,

$$b_e = 0.76\sqrt{rt}$$

where r is the radius of the shell and t, its thickness.

When the shell is only on one side it acts as *L*-section and the effective width is taken as $0.38\sqrt{rt}$.

The shear on the support is the reaction and is maximum in the middle of the stiffener and its value as shown in Figure 10.2[1] is,

$$T = T_\phi \int_0^{L/2} \sin\frac{\pi x}{L}\,dx = T_\phi\left(\frac{L}{\pi}\right) \quad \text{for a single shell on one side}$$

The reaction acts on the transverse with an eccentricity e = distance between C.G. of support and midpoint of shell section it supports. Hence, the effect is a direct tension and a moment. The moment *will produce tension at the bottom of the support* if the shell is placed below the C.G. of the support, and compression, if the shell is placed above the C.G. of the support.

10.2.1 Supports on Long Shells on *T* or *L* Beams

We design the support as a *T*-beam for continuous shells or *L*-beam for single shells for the following forces:

1. Load of the shell (1/2 load for simply supported shell)
2. Self weight
3. Tension and moment created by the shear from the shell on to the support.

10.2.2 Design of Supporting Frames

When the series of shells are supported on frames, we transfer the loads on the frames and design the frame using moment distribution method [2].

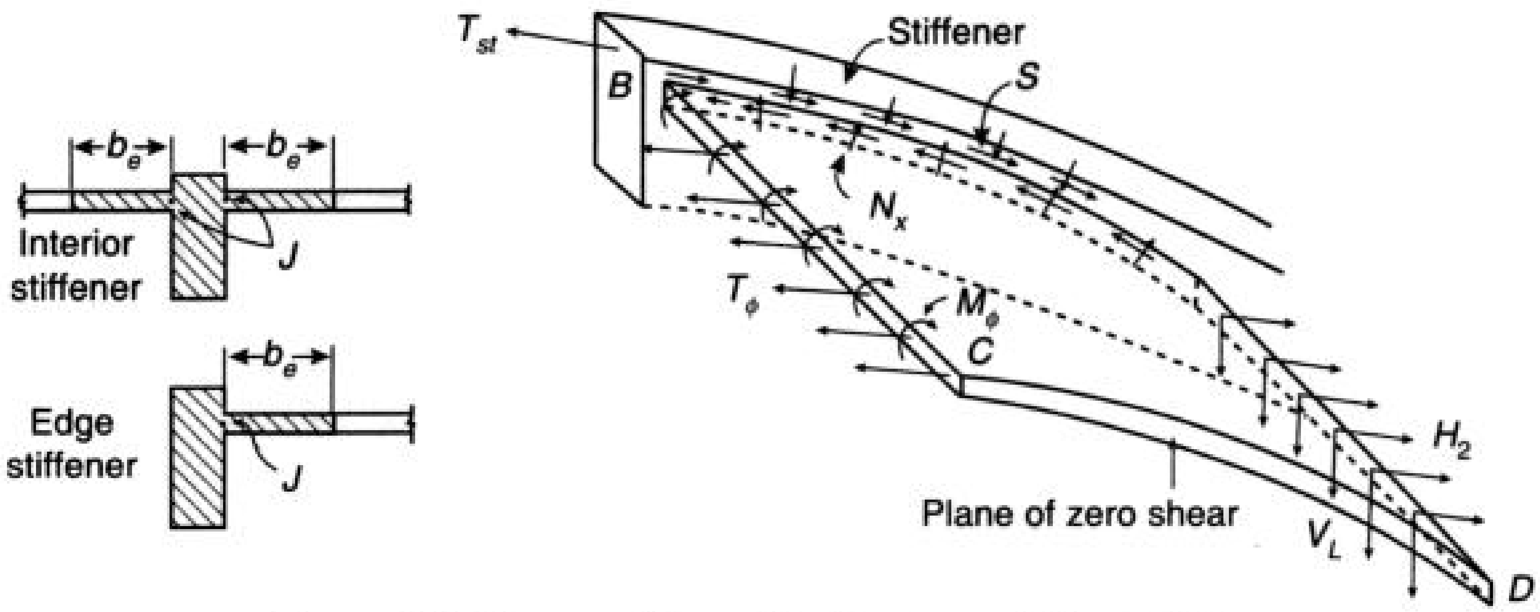


Figure 10.2 Forces acting on the transverse of a long shell.

10.3 DETAILING JUNCTION OF SHELL AND TRANSVERSE

Care should be taken in detailing the junction between the transverse and the shell. As already seen, in long shells the supports act as beams or frames. In short shells, these supports are usually arches. Detailing of support or long shells is shown in Figure 10.3 and explained in Example 6.1.

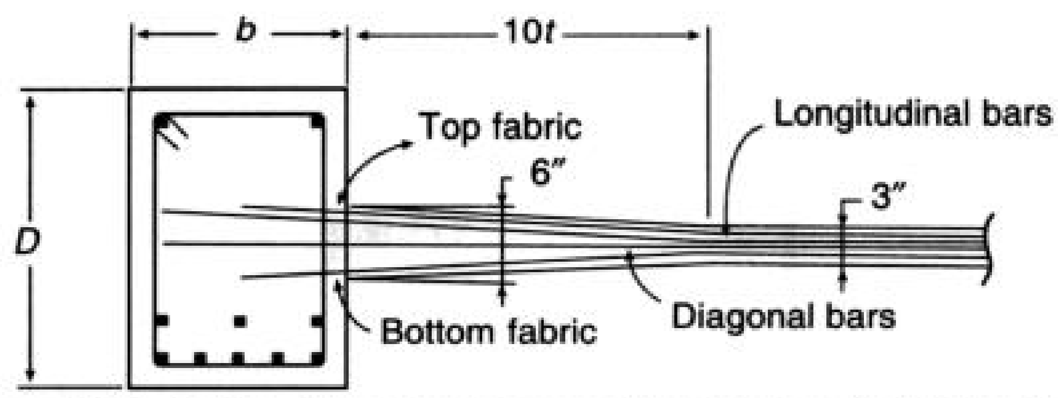


Figure 10.3 Section of transeverse support in circular cylindrical shell.

SUMMARY

Unlike the support of an ordinary beam, the transverse which supports a long cylindrical shell should be considered as part of the shell. It should be joined as an integral part of the shell. This chapter explained the forces that act on the support and its design.

EXAMPLE 10.1 Design of end transverse of a cylindrical shell

A cylindrical shell without edge beam is of length 15 m, thickness 75 mm, radius 7.5 m, chord length 9.65 m, $\phi_k = 40°$ and $h = 1.75$ m (Example 5.2). The value of T_ϕ at crown is assumed as 3000 kg/m. The shell is connected to the top of the diaphragm whose depth at the end is 40 cm and its width is 30 cm. (Assume Fe 250 steel with allowable tension as 1400 kg/mm^2.)

The following Figure E10.1 illustrates the details of the transverse

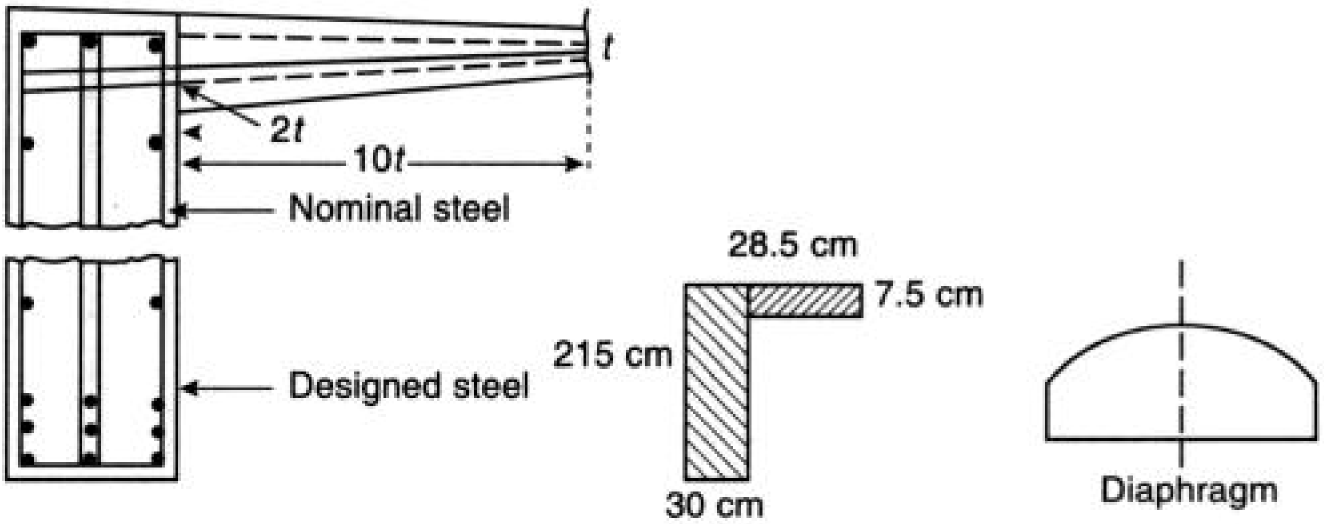


Figure E10.1

Reference	*Step*	*Calculations*
	1.	*Find the C.G. of the L-beam formed by the diaphragm at midsection*
Sec. 10.2		Max. depth of diaphragm = 175 + 40 = 215 cm, width = 30 cm
		Effective width of *L*-beam = $b_e = 0.38\sqrt{7.5 \times 0.075}$
		$= 0.285$ cm = 28.5 cm
		Taking moments about the top to find C.G. of section (in cm units), C.G. of *L*-section at centre
		$$= \frac{\left[(7.5 \times 28.5)\left(\frac{7.5}{2}\right)\right] + \left[215 \times 30 \times \left(\frac{215}{2}\right)\right]}{213 + 6450}$$
		= 104.2 cm = 1.042 m from top
Sec. 10.2	2.	*Total tension on diaphragm due to shear on diaphragm*
		T_ϕ at crown of the shell $\approx 3000 \sin\left(\frac{\pi x}{L}\right)$
		Tension,
		$$T = \int_0^{L/2} 3000 \sin\left(\frac{\pi x}{L}\right) dx = 3000 \times \left(\frac{L}{\pi}\right) = \frac{3000 \times 15}{3.14} = 14331 \text{ kg}$$
		Moment due to T about C.G. in step neglecting 1/2 thickness of shell.
		Moment = –14331 × 1.042 m = –14933 kg.m
		(This moment produces tension at the top of the diaphragm and compression at base.)
		[Note: If the CG of the transverse is above the shell, then tension will be produced at the bottom of the transverse.]

Reference	Step	Calculations
	3.	*Find dead and live load from shell to transverse*
		Circumferential length of shell, $r\theta = 7.5 \times \left(\frac{3.14 \times 40}{180}\right) = 5.23$ m
		DL per m^2 of shell = 0.075 × 2400 = 180 kg/m^2
		Assume Eq. dead load due to LL along shell = 120 kg/m^2
		Total load due to 15 m length of shell = 300 × 5.23 × 15 = 23600 kg
		(The moment due to this load produces tension at base of the transverse.)
	4.	*Find shear and bending moment due to DL + LL from shell resting on transverse*
		Shear = $\frac{23600}{2}$ = 11800 kg
		Calculate BM as if it is a simply supported beam.
		BM $\frac{wL^2}{8} = \frac{WL}{8} = \frac{23600 \times 9.65}{8}$ = 28467 kg.m
		(More refined calculations can be also made for the circular distribution.)
	5.	*Find BM and shear due to self weight of transverse*
		Area of the transverse = Rect. part + Circular part
		Rect. = 0.3 × 0.4 × 9.67 = 1.16 m^2
		Circular = Area of sector – Area of triangle
		$= \frac{\pi \times (7.5)^2 \times 40}{180} - \frac{1}{2} \times 9.65 \times 5.85 = 11\ \text{m}^2$
		Total volume = (11 + 1.16) × 0.3 = 3.648 m^3
		Total wt. = 3.648 × 2400 kg
		BM = $\frac{WL}{8} = \frac{3.648 \times 2400 \times 9.67}{8}$ = 10583 kg.m
		Shear = $\frac{3.648 \times 2400}{2}$ = 4376 kg
	6.	*Find total shear and design for shear*
		Shear = 11800 + 4376 = 16176 kg
		Section at end 30 × 40 cm
		Max. shear stress = $\frac{16176}{30 \times 40}$ = 13.48 kg/cm^2
		Max. shear allowable for M20 concrete is about 18 kg/cm^2.
		Design for shear as maximum allowed without shear reinforcement is only of the order of 5 kg/cm^2. Hence design for shear.

Reference	*Step*	*Calculations*
	7.	*Find total BM for design*
Step 2		BM due to eccentricity of T_ϕ = –14933 kg.m (compression at base)
Step 4		BM due to dead and live load on shell = 28,467 kg.m
Step 5		BM due to self wt. of transverse = 10,583 kg.m
		Total = 24117 kg.m
	8.	*Find direct tension and BM due to tension*
Step 2		Tension = 14331 kg along the centre of the shell.
Step 7		Positive BM = 24117 kg.m
	9.	*Design of reinforcements (section subjected to tension and moment)*
		As the negative BM due to tension is less than the +ve moment, we can place the tension steel below the N.A.
		A_s required for direct tension = 14331/1400 = 10.2 cm^2
		Provide 6 bars of 16 mm = 12.06 cm^2
Step 7		Let us place these bars below N.A. with an eccentricity of 9 cm.
		BM taken by these bars = 12.06 × 1400 × 0.9 = 15,195 kg.m
	10.	*Find steel for balance of the moment*
Steps 2, 7		Balance moment = 24117 – 15195 = 8922 kg.m
		Assume $jd = 0.86\ d$; d = 200 mm
		$A_s \text{ required} = \dfrac{8922\times 100(\text{kgcm})}{0.86\times 200\times 1400} = 3.71 \text{ cm}^2$
		(Provide 3 of 16 mm = 6.03 cm^2)
	11.	*Check percentage of steel provided*
		Total steel = 12 Nos. 16 mm = 24.12 mm^2
		$\% \text{ Steel} = \dfrac{24.12\times 100}{215\times 30} = 0.37$
		[**Note:** This is greater than 0.34% we usually provide in beams for Fe 250 steel in basis of *bd*. For Fe 415, the min. steel prescribed for beams is about 0.2%.]
	12.	*Continuity bars between shell and support*
Step 1		In general, all the diagonal shear reinforcements placed in the shell are anchored into the transverse. In addition, we may provide 6 mm bars (as 20 cm × 20 cm fabric) to a length of 75 cm (more than 2.5b_e where b_e = 28.5 cm) into the shell for anchorage of shell into the support.
		[**Note:** The junction between the transverse and the shell should be thickened as shown in Figure Ex. 10.1.]

REVIEW QUESTIONS

1. What are the types of support structures used to support the ends of (a) long cylindrical shell and (b) short cylindrical shell?
2. Indicate the forces that act on the end transverse stiffeners of long cylindrical shells.
3. What will be the arrangement of the transverse for a series of multiple sell for a factory?
4. Give the layout of a multiple shell assembly with the shell (a) supported on top of the stiffener and (b) supported at the bottom of the stiffener.

REFERENCES

[1] ASCE Manual No. 31, American Society of Civil Engineers, 1952.
[2] Design of Barrel Shell Roofs, Portland Cement Association, USA, 1954.

11

DESIGN OF PARABOLOID SHELLS (SHELLS FORMED FROM TWO PARABOLAS)

11.1 INTRODUCTION

In this chapter, we briefly examine paraboloid shells. Paraboloid shells are translational shells formed from two parabolas placed at right angle to each other and by the translation of one parabola over the other. The common paraboloids may be subdivided into the following three groups:

1. *Elliptical paraboloid*. This type is formed by two unequal parabolas with same curvature sign (both convex curved in the same direction).
2. *Circular paraboloid*. This type is formed by two equal parabolas with same curvature sign (both concave curved in the same direction).
3. *Hyperbolic paraboloid*. This type is formed by two parabolas of opposite curvatures (one convex and the other concave curved in the opposite directions).

They derive their name according to the type of curve obtained by the intersection of the shell surface with an arbitrary horizontal plane z = constant, as shown in Figure 11.1 and Table 11.1.

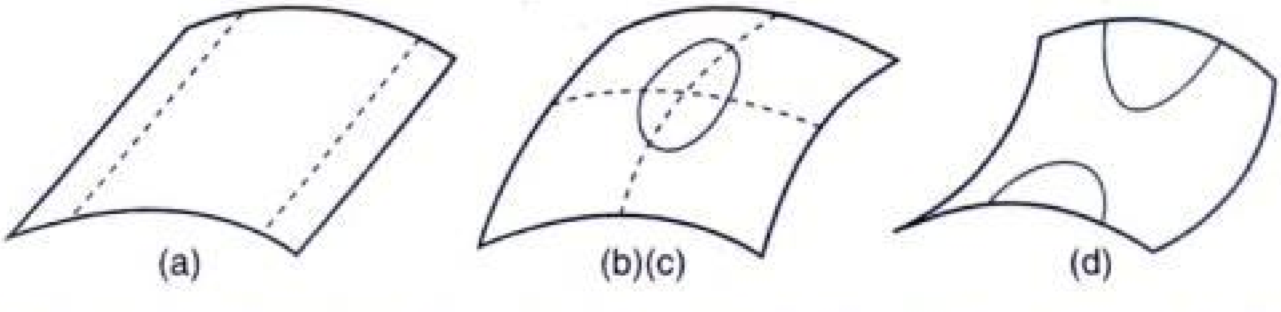


Figure 11.1 Traces formed by intersection of a horizontal plane with different shell surfaces: (a) Straight lines with a cylindrical shell, (b) Ellipse (or circle) with an elliptical (or circular) paraboloid shell, and (c) Hyperbolas with a hyperbolic paraboloid shell.

TABLE 11.1 Types of shells formed by parabolic directrix [Fig. 11.1]

Directrix	*Generator*	*Horizontal Section*	*Name of Surface*	*Figure 11.1*
Concave Parabolas	Straight Line	Straight Line	Parabolic Cylinder	(a)
Convex Parabola	Unequal Convex Parabola	Ellipse	Elliptical Paraboloid	(b)
Convex Parabola	Equal Convex Parabola	Circle	Circular Paraboloid	(b)
Convex Parabola	Concave Parabola	Hyperbola	Hyperbolic Paraboloid	(c)

As shown in Figure 11.1(a), in a parabolic cylindrical shell, the intersection with a horizontal plane will be straight lines. When the translational surface is made up of two *unequal parabolas* (with the same curvature sign), the curve obtained by a cut by a horizontal place will be an ellipse and hence it is called an *elliptical paraboloid*. If the parabolas are *equal*, the curve obtained by the intersection of a horizontal plane will be a circle and so it is called a *circular paraboloid*. On the other hand, the negative Gaussian surface got by the translation of two parabolas with opposite curvatures one moving on the other by the horizontal plane will be a hyperbola. So, such a shell is called a *hyperbolic paraboloid*. By change of axis this hyperbolic paraboloid can be also looked upon as a warped surface or a *doubly ruled surface* (refer section 2.2.3). As the formwork needed to form this ruled surface is simple, it is a new form of shell roof construction very popular among modern builders.

References [1] to [7] give the theoretical part of the analysis of hyperbolic paraboloid shells. Instead of going through the intricate mathematical analysis, we focus only on the basic static principles which are easy to understand. It is very easy to design a hyperbolic paraboloid and if we detail them carefully, they are found to perform well in the field also. In this chapter, we deal mainly with hyperbolic paraboloid (also called *hypar*) shells, with a short introduction to elliptic paraboloids. (For equations of curves, see Appendix C.) The shell part is most subjected to shear producing tension and compression.

11.2 TYPES OF HYPERBOLIC PARABOLOIDS

Principally, we get two types of hyperbolic paraboloids, as shown in Figure 11.2. First is the *one bounded by curved lines*. It is a simple surface obtained by one parabola moving over another parabola. They are bounded by curved lines. The second type *bounded by straight lines* (with edges parallel to generators) is got by change of axis. In this type, we get a *ruled surface* (or a warped surface) bounded by straight lines as shown in Figure 11.2.

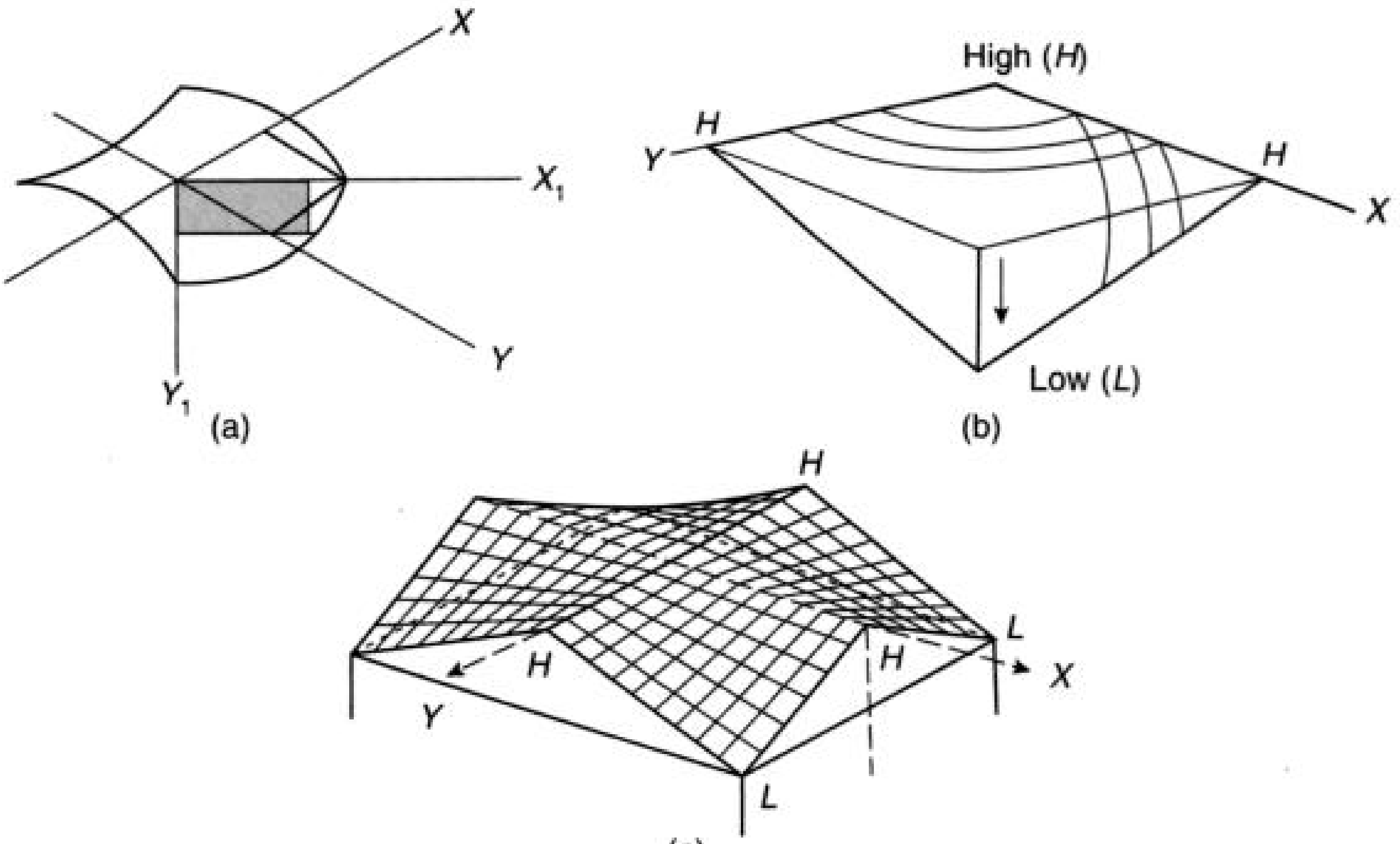


Figure 11.2(a) Formation of hyperbolic paraboloid with straight edges: (a) Change of coordinates, (b) Warped surface which is a ruled surface of hyper shell, and (c) Hipped roof formed by four hyper shells (letter *H* denotes high and letter *L* low levels).

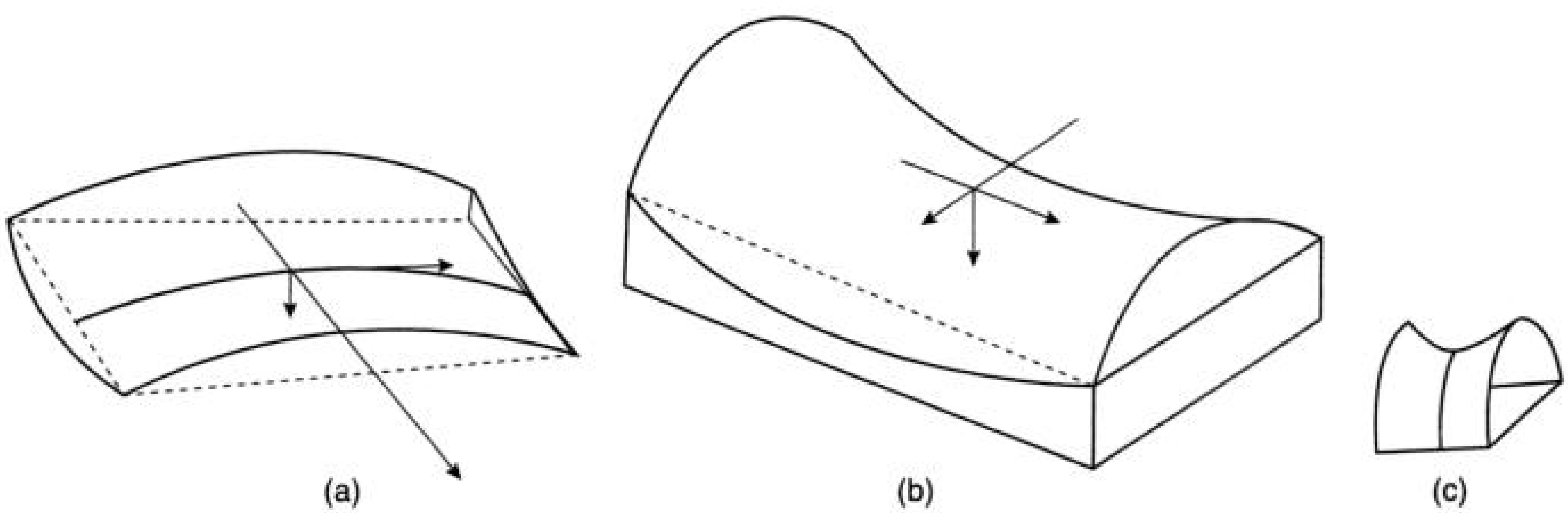


Figure 11.2(b) Hyperbolic paraboloid roofs with curved edges formed by translation of two parabolas with curvatures of opposite signs with different drainage patterns.

The first type of curved shells is commonly used as precast large span roof elements. It can be prestressed, if necessary, for use in long spans. They are extensively used to cover long span buildings like storage sheds, factories, etc. A series of such prestressed shells 20 m in span and 2.5 m width were used for a factory in Bangalore. The second type is a ruled surface with straight edges. Because of its ease in construction of formwork, it is nowadays very popular for large span buildings such as assembly halls and exhibition halls. As these are the most commonly used hypar shells, we examine the second type in more detail. As we will see in Section 11.9, these hypar shells can be rectangular or parallelogram in plan. The latter are called *oblique hypar shells* (See Figures 11.8 to 11.10).

11.3 EQUATION OF HYPAR SHELLS WITH STRAIGHT RECTANGULAR EDGES

A much easier way than the mathematical way to define the surface of a hypar shell with straight edges is to consider how it can be physically formed as a ruled surface. This is shown in Figure 11.3. Let *OABC* be a plastic sheet of size $a \times b$. We depress (or elevate) the edge B to a depth (height) h. Then, the surface is defined by the equation

$$z = \left(\frac{h}{a}x\right)\left(\frac{1}{b}y\right) = \left(\frac{h}{ab}\right)xy = (c)xy$$

or, simply, by

$$\left(\frac{hxy}{ab}\right) \tag{11.1}$$

The value of $c = h/ab$ is called the *twist* or *warp*. It denotes the difference of slope between the generators of unit length. If the value of h is small, less than 1/5 of the larger side, *it is a shallow shell*. If the twist is large, it is classified as a deep hypar shell.

[**Note:** In these rectangular figures, three corners are at the same level and the fourth is either above or below the others. See also Section 11.5.]

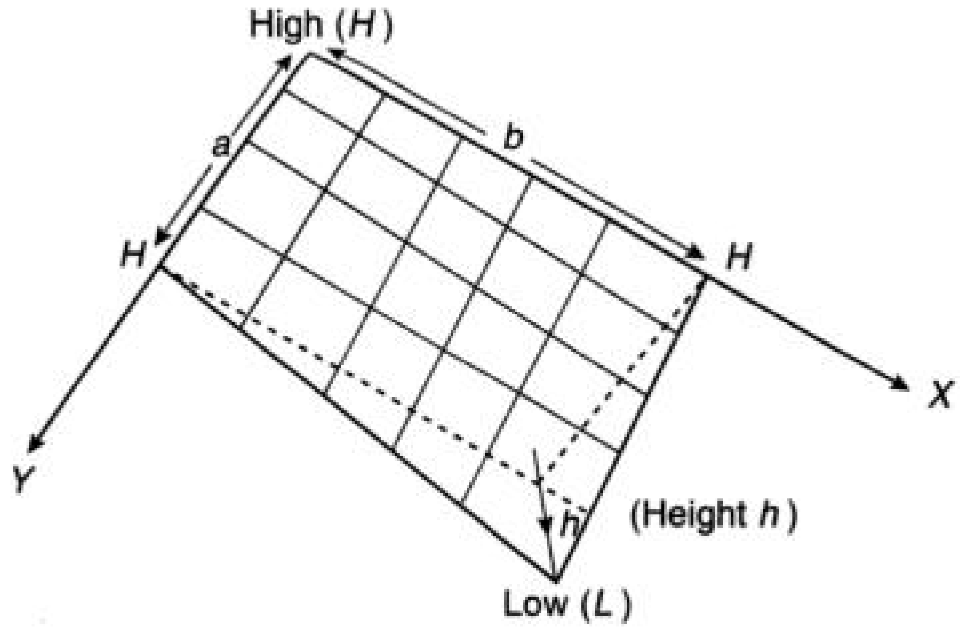


Figure 11.3 Derivation of equation for a hyperbolic paraboloid (hyper) shell with straight edges.

11.4 TYPES OF H.P. SHELL ROOFS WITH STRAIGHT EDGES

Different types of roofs can be formed from HP shells with straight edges. Some of them are shown in Figure 11.4. It should be noted that the whole structure is made of a number of separate hypar shells. For example, all the shells shown in Figure 11.4 are made of four individual hypar shells and each is joined to the other by edge beams. Hipped hypar (shown in Figure 11.4(c)) designed for span 20 m × 20 m is quite common. In all these shells, the coordinates are at right angles. (As already stated, hypar shells with oblique axes called *oblique hypar shells* are described in Section 11.9. Rectangular hypar shapes can be also used as foundations as described in Reference [8].)

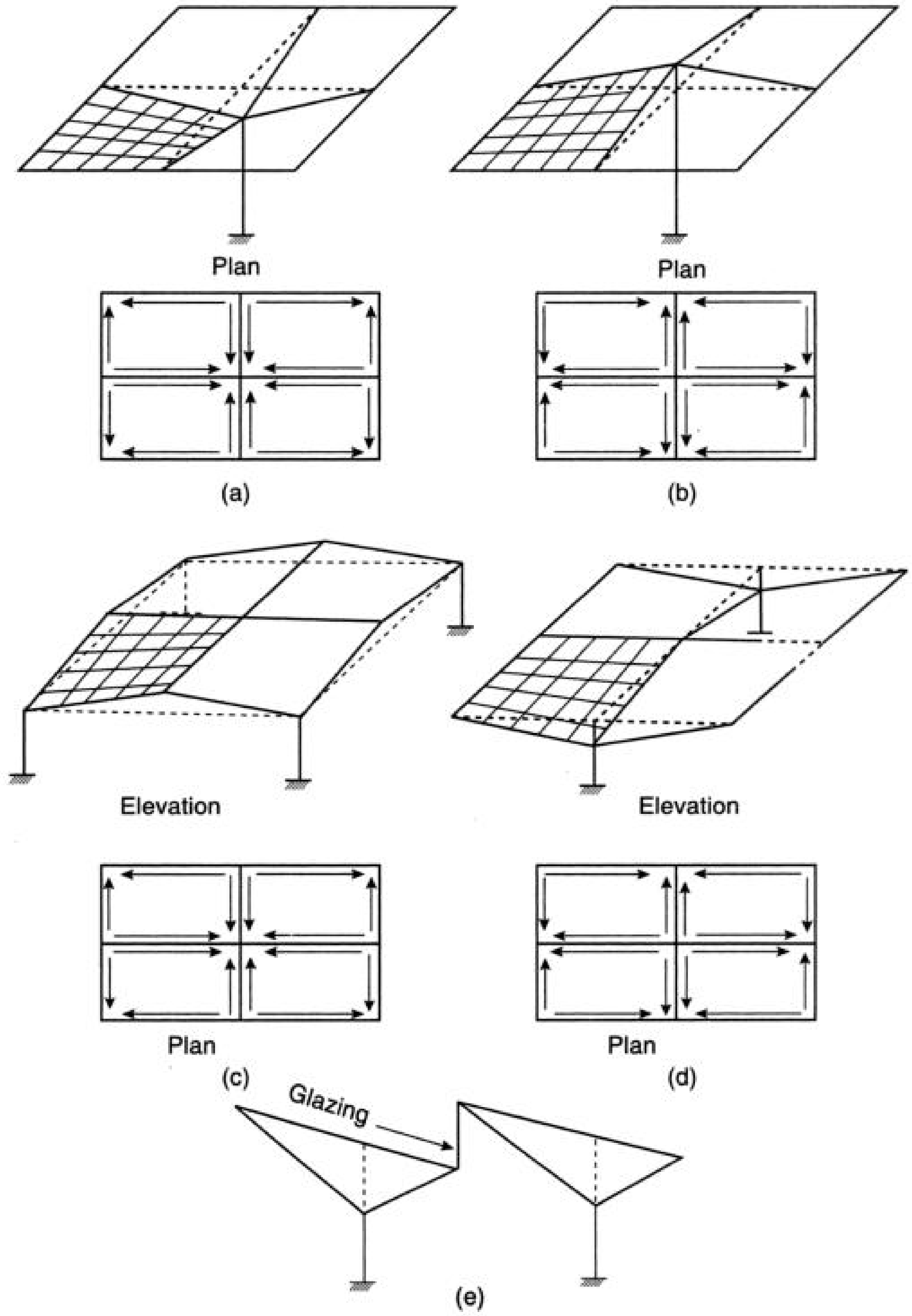


Figure 11.4 Different types of shell roofs, each with four hypar shells with straight edges: (a) Inverted umbrella roof, (b) Umbrella roof, (c) Hipped roof on four supports, (d) Cantilever type roof on two supports, and (e) Tilted umbrella roof for improved lighting. [**Note:** Arrows show flow of forces on edge members.]

11.5 SHALLOW AND DEEP H.P. SHELLS

When the height of uplift of the corner is not too large, we can assume the approximations that the dead load of the shell is the same as a uniform load on the plan projection of the

roof. Such shells are called *shallow shells*. It can be proved that H.P. shells, *where rise is equal to or less than 1/5 of the longer side of the shell*, can only be considered as shallow [1]. Otherwise, it is called a *deep hypar shell*. We restrict out study to shallow shells. (For theory of analysis of high rise hypar shells, see Reference [1].)

11.6 ANALYSIS OF THE SHELL PART OF SHALLOW HYPAR SHELLS WITH STRAIGHT EDGES

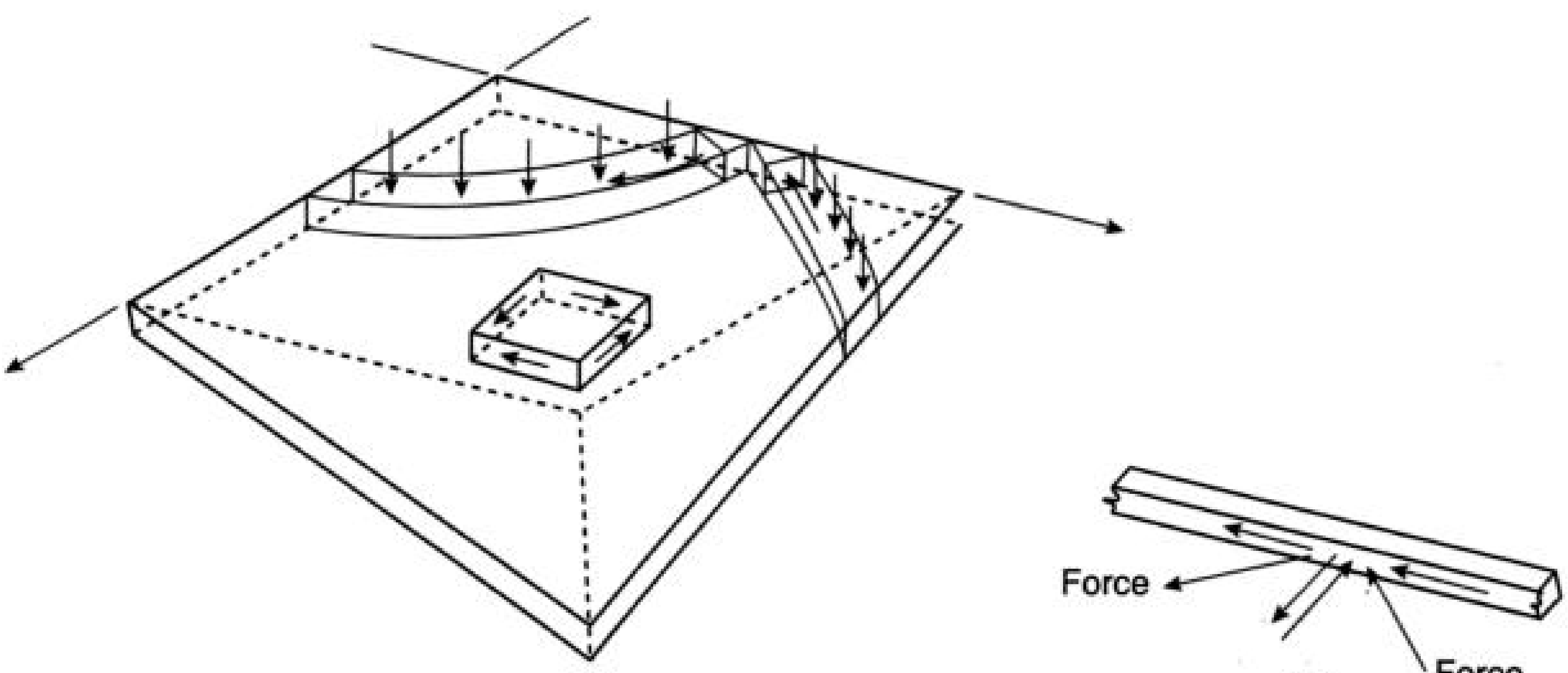
The following analysis is true for *shallow hypar shells with straight rectangular* edges. For deep hypar shells, the shear along edges is not uniform. The simple analysis is based on arch action.

Figure 11.5 shows the parabolic arches which are perpendicular to each other and with which the hypar shell surface can be assumed to be made of. We may assume that these arches take the loads on the shells. One will be in tension and the other in compression. If we reverse the theorem that pure shear produces pure tension and compression, the resultant of these forces can be taken as pure shear in the shell proper. We have to provide steel for the tension along the line of tensile arch. But as it is inconvenient, we provide equal steel in both the directions of the edge members. Thus,

$$\text{Shear} = \text{Horizontal thrust of each arch}$$

$$= \frac{w}{2}\left(\frac{ab}{h}\right) = \frac{w}{2c} = \frac{w}{2(\text{warp})} = \frac{wR}{2} \text{ or } \frac{wab}{2h} \tag{11.2}$$

where $\frac{h}{ab}$ is called the warp and the reciprocal $\frac{ab}{h}$ is the radius of curvature R. (It is easy to remember this formula for shear in the shell as $\frac{wab}{2h}$.)



(a) (b)

Figure 11.5 Forces acting in a hypar shell: (a) Forces in the shell, and (b) Forces on edge members resolved into components resulting in shear only.

Equation (11.1) gives the tensile or compressive thrust induced in the shell by a uniform load on the shell. The shell needs to be reinforced only for this force. In reality, since the slope of the surface (h/ab) steepens near the column supports, the load cannot be considered as uniform, but this departure for a shallow shell can be tolerated. The designer should be aware of this aspect. The pure tension and compression produces only shear in the shell proper as shown below.

$$\text{For a H.P. shell, } N_x = N_y = 0 \text{ and } N_{xy} = \frac{wab}{2h} \tag{11.2a}$$

We can identify the direction of shear in the shell and the forces in the edge members by identifying the tension and compression parabolic arches. The entire load is taken by the shell by pure shear, which indirectly produces equal compression and tension of the same magnitude in the edge members. The total force in the edge members is the summation of all the shear forces.

The value of the radii of curvature $\dfrac{ab}{h}$ is also used to find the slenderness of the shell.

$$\text{Slenderness} = \lambda = \frac{t}{R} = \frac{th}{ab} \text{ (where } t \text{ is the thickness of the shell)} \tag{11.3}$$

In practical shells λ varies from $\dfrac{1}{500}$ or (0.002) to $\dfrac{1}{250}$ or (0.004). For the umbrella type of hypar shells, the usual value of λ adopted is 0.004. The larger is the value of λ, the more rapid is the decrease of the secondary stresses due to bending from the corners for the umbrella shells [2].

11.7 ANALYSIS OF THE EDGE MEMBERS

It is very important that we should design the edge members with care. Each hypar shell must have edge stiffening members along all the four sides. The shear in the shell has to be *balanced by equal and opposite forces in the edge members*. The edge member *progressively collects* the shear forces and transfers it to the supporting members.

The direction of the forces in the edge members can be easily drawn on the following principle as shown in Figure 11.4. Consider each H.P. separately taking the load as acting downwards. The flow of load in the edge beams connected to the column is first drawn by arrows from the *high to the low point*. In the adjacent sides of the above sides, the arrow is drawn (as in the case of drawing shear forces) away from the corner they meet. (This will mean that in the opposite side the arrow showing the force will point in the opposite direction.) For shallow shells, we can also assume the following rule which is not exactly true for deep shells:

Magnitude of force in edge member from each shell

$$= \text{(Shear in the shell) per unit length} \times \text{(Length of member)} \tag{11.3}$$

It is important to note that if we have shells on both sides of the edge member, it collects shear from both sides. We will illustrate the above procedure by Example 11.1. Tensile members must be properly designed. In compression members, we should provide a minimum of 0.6 percent steel.

11.8 SUPPORTING DEAD WEIGHT OF EDGE MEMBERS

In the above analysis, we have considered the effect of only the dead and live loads on the shell. We have not made any provisions for supporting the edge beams of the shell. Hence, the weight of the assembly of edge members should be designed to be carried by the structure independently of the analysis of the loads on the shell. This aspect is very important in the design of these shells.

As an example, we can take the umbrella roof shown in Figure 11.4, when the edge members are in tension. In these edge beams, especially near the corners at the far end from the column, the beam action is predominant. The edge beams act as cantilevers for a small part of their length from the corner. Hence, these edge members should be provided with negative steel. This self-supporting design also should be carried out in all field projects. It is not included in the examples given at the end of this chapter as it can be carried out by conventional methods. (A discussion about this is given in Reference [1]).

11.8.1 Detailing of Steel in Hypar Shells

The detailing of the shell proper is easy. We provide reinforcement in two directions (in the directions of the edges in rectangular slabs) as shown in Figure 11.7 and Example 11.1. The detailing of edge beams and columns is also very important. In many instances a trouble in the performance of these shells occurs due to inadequate detailing of the edge members and supporting members. As examples of detailing, the following two types of shells are given below:

1. Detailing of hipped roofs: The detailing of hipped roofs is shown in Figure 11.6. The edges of the shell are generally thickened over a length $b_e = 3.38\sqrt{Rt}$ and reinforcement provided at top and bottom to *act as a frame*. ($R = ab/h$ as shown in Section 11.3.) Alternatively the design of the edge beams as a frame to carry its own weight can be planned. For more information, the reader may consult Reference [2].]

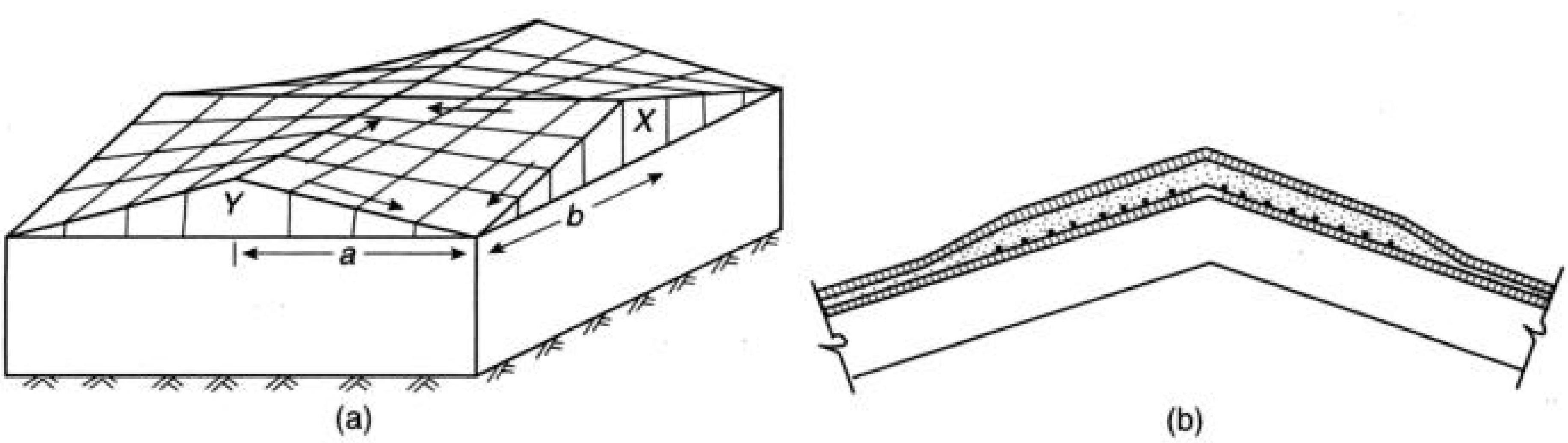


Figure 11.6 Formation of a hipped roof: (a) Shape of roof and flow of forces, and (b) Thickening of edges of shell and position of edge members. (See also Fig. Ex. 11.2.)

2. Detailing of umbrella roofs: Figure 11.7 and Example 11.1 give the method of detailing umbrella roofs made of hypar shells. [For more details, consult Reference 1.] In smaller shells, instead of beams, the *edges can be thickened* to give beam effect (Figure 11.7).

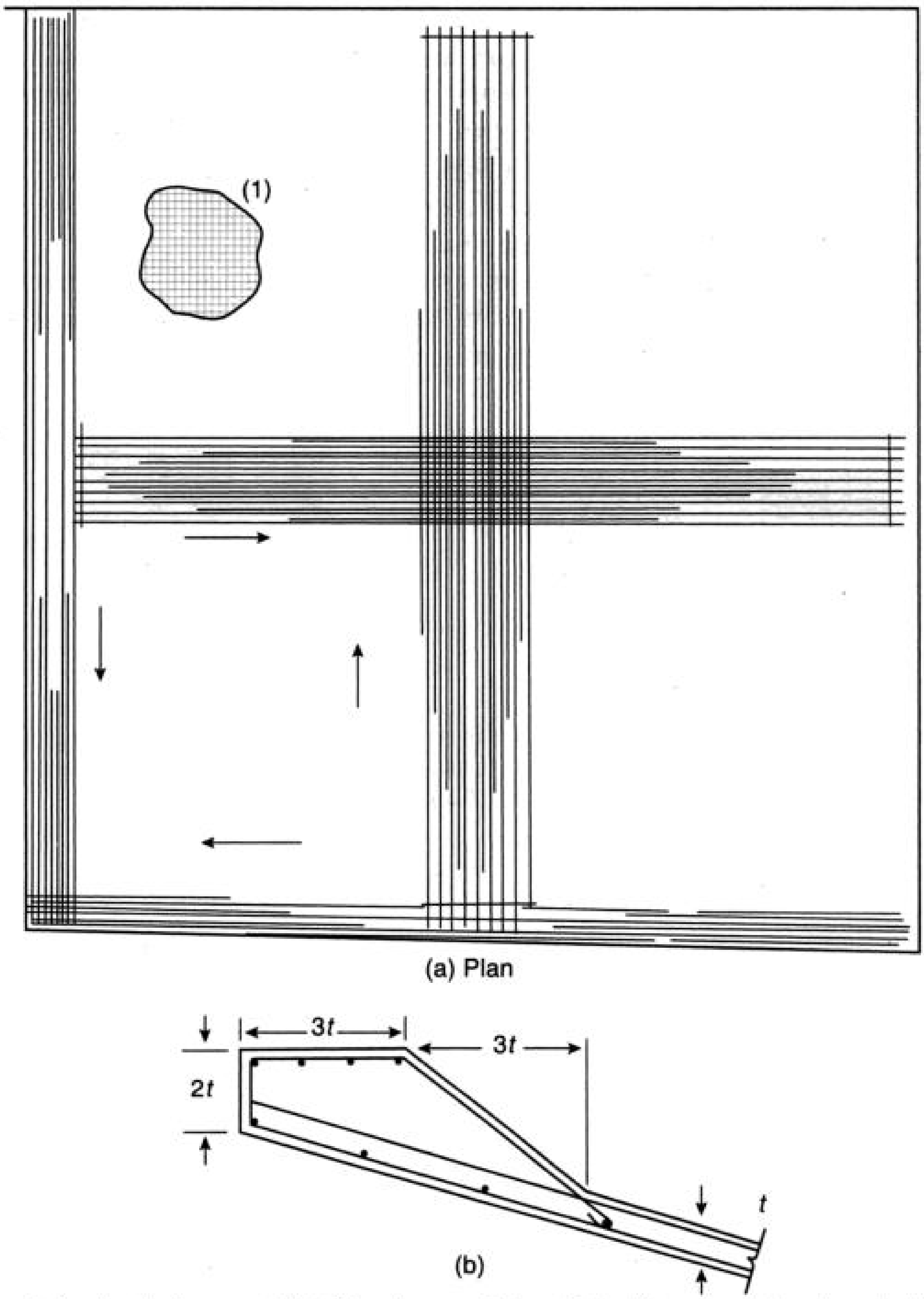


Figure 11.7 Inverted umbrella hypar roof: (a) Plan shows curtailment of reinforcements in edge and ridge members (1) Shell reinforced both ways with central layer of steel or two layers one on top and the other on bottom of shell, and (b) Formation of edge member by gradual thicknening of shell as an alternative to rectangular beams.

11.9 OBLIQUE HYPAR SHELLS

We can also construct hypar shells with straight edges over a *parallelogram shaped area* with the angle at origin as ϕ instead of 90°. They are called *oblique hypar shells*. Such shells

can be used to cover a rectangular hexagonal plan, as shown in Figure 11.8, with three columns. The three inner edge members are horizontal and in compression while the six peripheral members are inclined and in tension. They are supported on three columns at three corners and the other three unsupported corners are at a depth h from the centre, as can be seen from the flow of forces in Figure 11.8. (Reference [1] gives the theory of the analysis of the shells with worked out examples.) Example 11.3 gives the design of such a hexagonal roof with these oblique shells.

We can also use oblique hypar shells, as shown in Figure 11.9 and as described in Reference [2], supported on four pillars with the centre and corners rising at a height h. Figure 11.10 shows another oblique shell described in Reference [3].

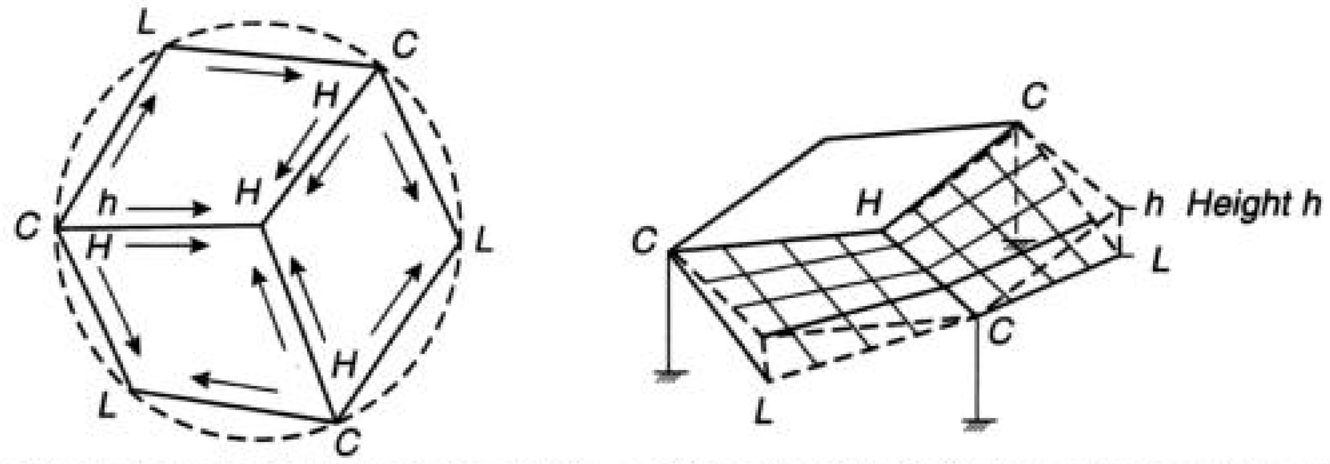


Figure 11.8 Oblique hypar shell on regular hexagonal plan on there supports Peripheral edge members, at inclined to low level, are in tension and inner edge members at high level are in compression (H = High level, L = Lower level, and C = Column). (See also Fig. Ex. 11.3.)

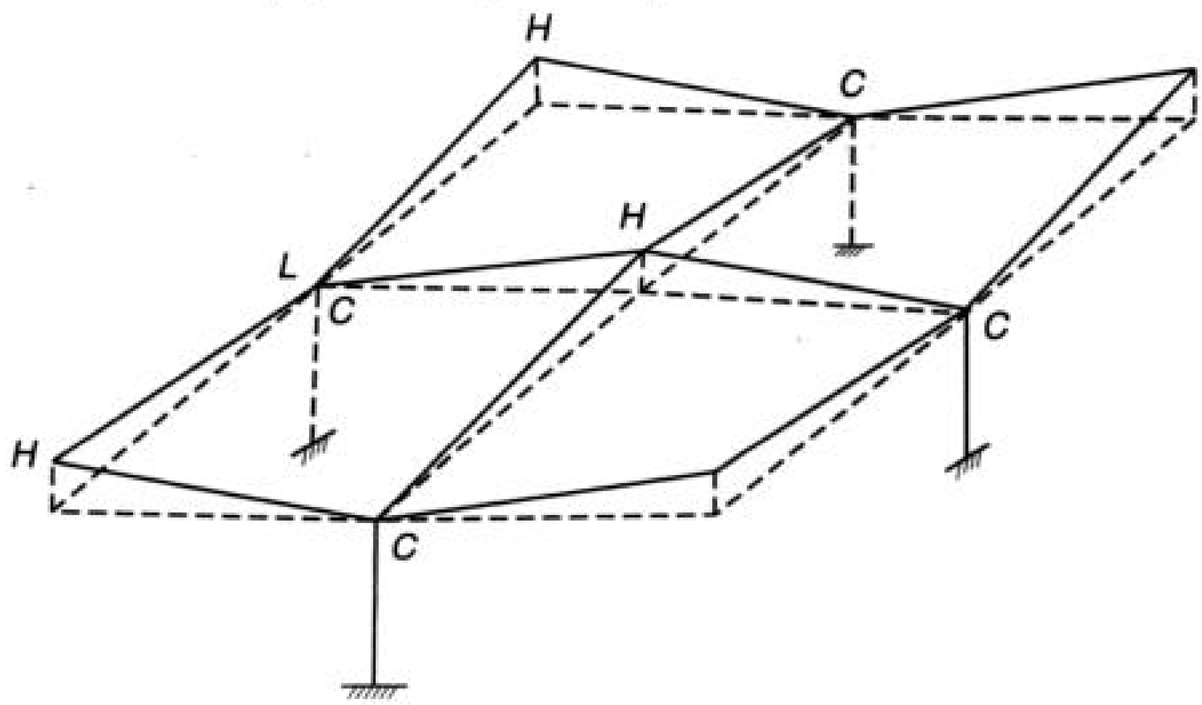


Figure 11.9 Oblique hypar on four supports (Turkey shed).

11.10 ELLIPTICAL AND CIRCULAR PARABOLOIDS

As already pointed out in Section 11.1, elliptical paraboloids are formed by two unequal parabolas both pointing downwards (with same curvature sign) and circular hyperboloids are formed by two equal parabolas pointing downwards (Figure 11.11(a)). These shells are commonly used to cover large spaces. For example, the tennis court at Wimbledon is

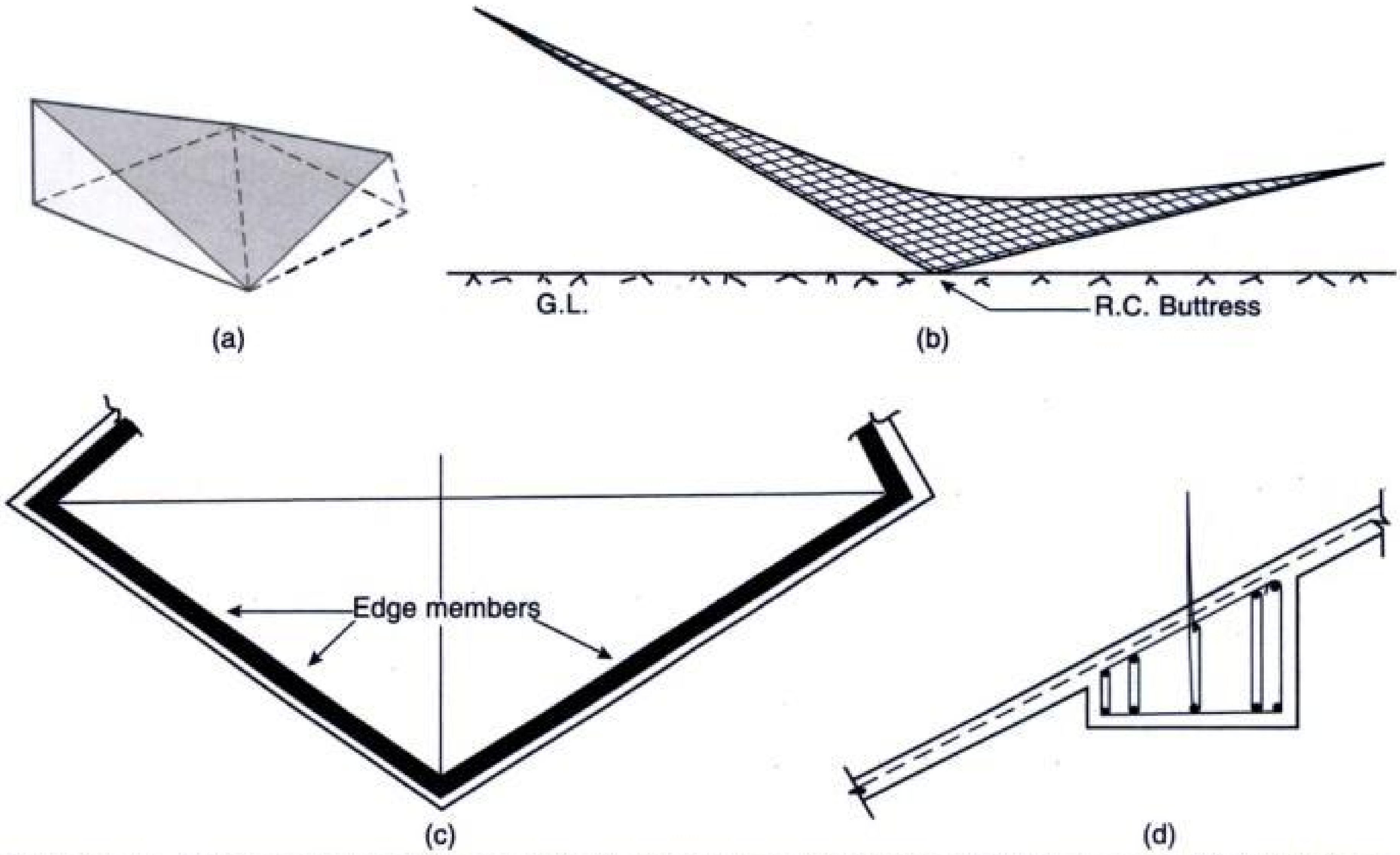


Figure 11.10 Oblique hypar used for a church: (a) General view, (b) Elevation, (c) Half plan, and (d) Detail of edge member.

covered by a circular paraboloid of the shape shown in Figure 11.11. A view of the roof of a factory formed with these shells is shown in Figure 11.11(b). (References [1], [2], [4], [5], [6] and [7] deal with this subject in more detail.)

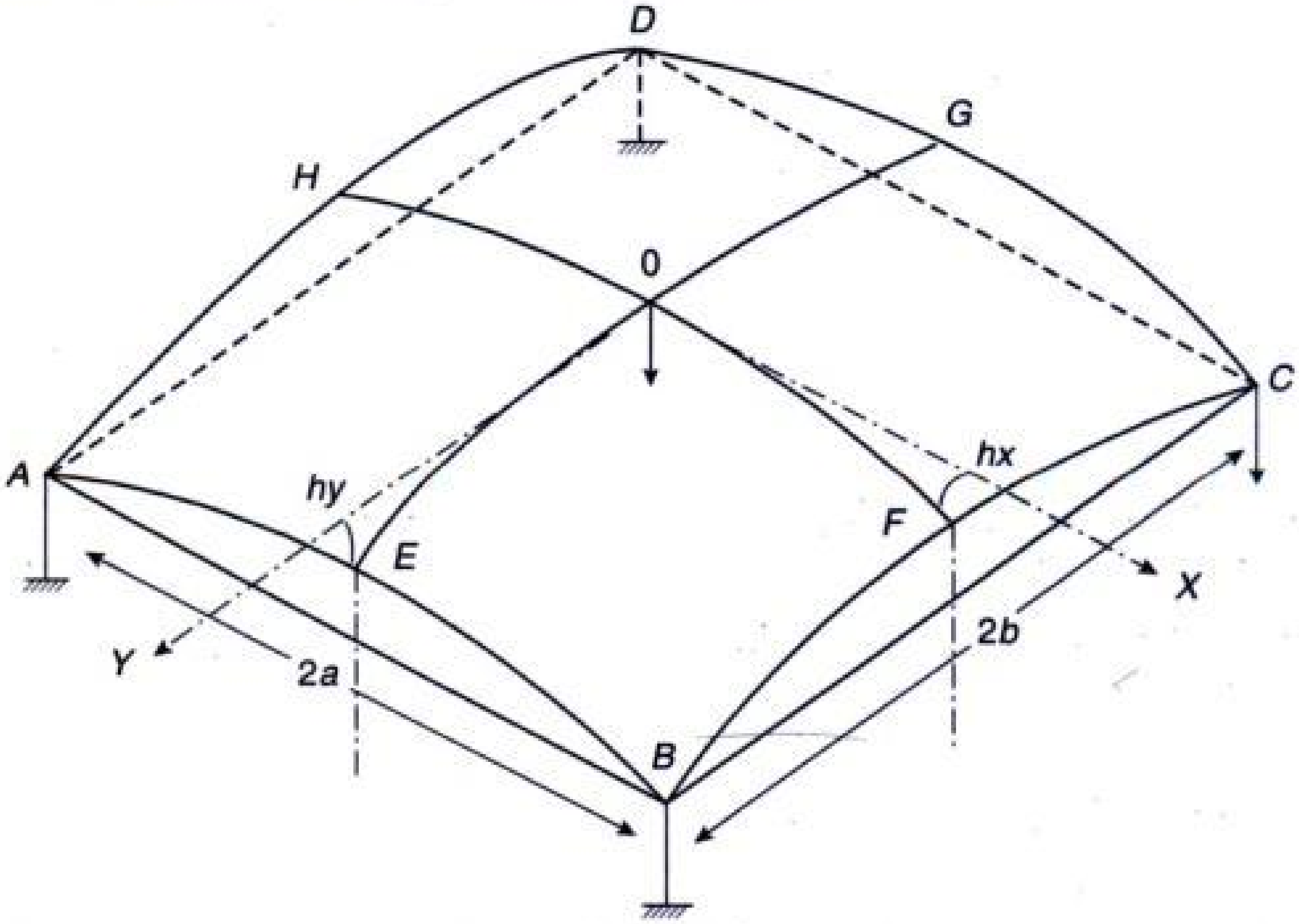


Figure 11.11(a) Example of an elliptical paraboloid roof formed by translation of two unequal parabolas *HF* and *EG* with curvature in the same direction. (See Appendix C, Figure C3.)

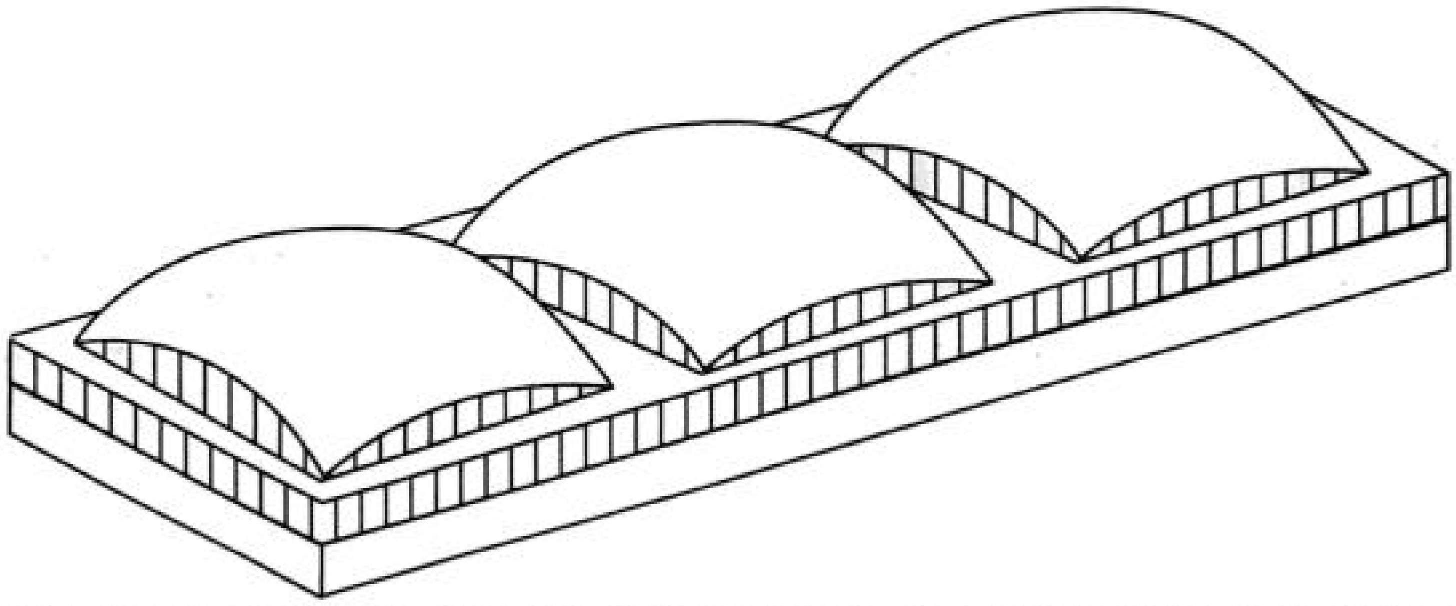


Figure 11.11(b) Example of circular paraboloid roofs formed by translation of two equal parabolas (with curvature in the same direction) used for a factory roof.

[It is interesting to note that the roofs of a few of the old North Indian temples built of bricks around 17th century are somewhat of this shape.]

The equation to the elliptic paraboloid with reference to Figure 11.11 is as follows:

Equation to parabola H.F. is $z_x = h_x\left(\frac{x}{a}\right)^2$

Equation to parabola E.G. is $z_y = h_y\left(\frac{y}{b}\right)^2$

Equation to elliptical paraboloid is $z = h_x\left(\frac{x}{a}\right)^2 + h_y\left(\frac{y}{b}\right)^2$ (11.5)

Equation for circular paraboloid is $z = \frac{h(x^2 + y^2)}{a^2}$ (11.6)

Equation for hyperbolic paraboloid with curved edges is

$$z = h_x\left(\frac{x}{a}\right)^2 - h_y\left(\frac{y}{b}\right)^2 \tag{11.7}$$

11.11 ACTION OF ELLIPTICAL PARABOLOIDS

We will briefly examine the geometry of the commonly used elliptical paraboloid shells and the forces that act on them. Details can be obtained from References [1], [5], and [6].

11.11.1 Shallow Elliptical Paraboloids

In practice, we use shallow elliptical paraboloids/shells. They can be taken as shallow if the total rise of the crown from the corners ($h_1 + h_2$) is less than 1/5 the diagonal of the rectangle of dimension ($2a \times 2b$) as shown in Figure 11.11. Thus the condition is that,

$$(h_1 + h_2) \le \frac{2}{5}\sqrt{(a^2 + b^2)}$$

In practice, we usually choose dimensions so that

$$\frac{a+b}{10} \le h_1 + h_2$$

Thus, we adopt $h_1 \ge \left(\frac{a}{10}\right)$ and $h_2 \ge \left(\frac{b}{10}\right)$. Usually, we adopt $\frac{1}{10}$ of the larger side (or $\frac{1}{20}$ th of the larger half side).

11.11.2 Curvature and Radius

By definition, curvature $K = \frac{1}{R_X} = \frac{\frac{d^2z}{dx^2}}{\sqrt{1+\left(\frac{dz}{dx}\right)^2}}$

(Neglecting $\left(\frac{dz}{dx}\right)^2$) = $\frac{d^2z}{dx^2}$ [let $h_x = h_1$ and $h_y = h_2$]

This gives, $\frac{1}{R_1} = \frac{2h_1}{a^2}$ and $\frac{1}{R_2} = \frac{2h_2}{b^2}$

or $R_1 = \frac{a^2}{2h_1}$ and $R_2 = \frac{b^2}{2h_2}$

Thus, for example, if we are planning an elliptical hyperboloid for a plan area $2a = 18$ m and $2b = 24$ m, we can adopt the following dimensions for a shallow shell:

Assume $h_1 = h_2 = \frac{b}{10} = \frac{12}{10} = 1.2$ m

Also, $R_1 = \frac{9 \times 9}{(2 \times 1.2)}$ = 33.75 m

and $R_2 = \frac{12 \times 12}{(2 \times 1.2)}$ = 60.0 m

11.11.3 Nature of Variation of Membrane Forces

As the shell is symmetric about the centre lines, we need to calculate the forces only one quarter of the shell using a grid with equal divisions on each side.

It will be found that the membrane forces are mostly shear. They are low at the central portion starting from zero at the centre and increase towards the edges. Thus, they are maximum at the corners.

At the boundary, these shells are generally provided with edge gable walls or other members of various types like arches with ties, portal frames to which these edge shears are transferred. However, as there is discontinuity at the edges, it also produces bending at these places. It is convenient to use published tables for design: table of coefficients for computing stress resultants in elliptical paraboloids. Such tables were published by Parme [6]. It is also available in Reference [7]. Practical designs can be easily made with use of these tables.

SUMMARY

Hypar (hyperbolic paraboloid) shells with straight edges are ruled surfaces. They are easy to design and also simple to construct. In this chapter, we dealt mainly with the analysis of stresses in shallow hypar shells with straight boundaries at right angles. (Design of deep hypar shells and oblique hypar shells are explained with examples in Reference [1].) Elliptical and circular paraboloids are also useful shapes to cover large areas. It is very important that the border members and the framework formed by the edge members should be designed as self-supporting structures.

EXAMPLE 11.1 [Design of inverted umbrella roof with one central pillar]

An umbrella roof as shown in Figure Ex. 11.1 below is to consist of four H.P. shells of sides $a = b = 7$ m assembled to cover 14 m × 14 m as one unit. Assume the shell thickness is 70 mm and the total UDL on the shell is 2.2 kN/m^2 [1].

[**Note:** The flow of forces in the edge beams can be easily drawn as described in Section 11.7.]

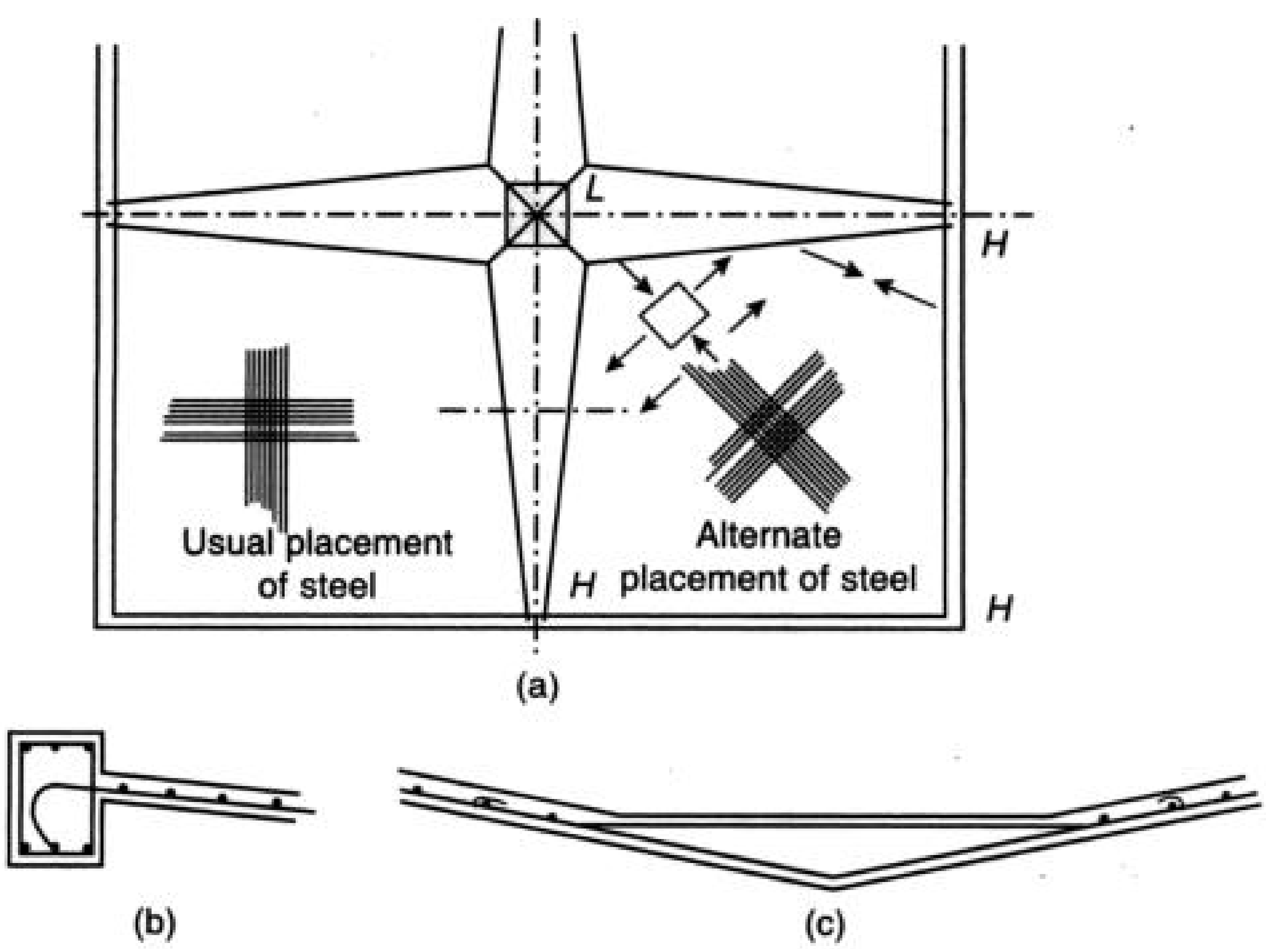


Figure E11.1 Plan and details of an inverted umbrella roof made with hypar shells: (a) General arrangement, (b) External edge beam, and (c) Internal edge beam. (See also Fig. 11.2b.)

Reference	*Step*	*Calculations*
	1.	*Estimate rise of shell assuming low rise shell*
		For a shallow shell, rise < 1/5 longest side of H.P. unit
		Rise = 7/5 = 1.4 m, Slope to horizontal = $\tan^{-1}(1.4/7) = 11.3°$
Eq. (11.3)		Slenderness ratio = $\frac{th}{ab} = \frac{70 \times 1400}{7000 \times 7000} = \frac{1}{500} = 0.002$
	2.	*Design of shell proper*
Eq. (11.2)		Shear = Normal force $= \frac{w}{2}\left(\frac{ab}{h}\right) = \frac{2.2 \times 7 \times 7}{2 \times 1.4} = 38.5$ kN/m
		Tensile or compressive stress is equal to the shear in the shell.
		Stress = $\frac{38.5 \times 1000}{70 \times 1000} = 0.55$ N/mm^2 (less than $\frac{f_{ck}}{10}$)
		Reinforcement required per metre for tension
		$= \frac{38.5 \times 1000}{\text{Allowable stress (180)}} = 213$ mm^2/m
		Though, ideally, it is to be placed along the tensile parabola, we will provide a mesh parallel to the sides in both directions.
		[**Note:** As regards the edge members from flow of forces described in Section 11.7 and Figure 11.3, we can find that the peripheral edge members are in tension, and central edge members are in compression.]
	3.	*Design of peripheral edge member (horizontal) (This is in tension)*
		Shear acts only on one side.
		Total tension = 38.5 × 7 = 269.5 ≈ 270 kN
		This member should be stiff. Let depth (d) ≈ $4t$, breadth (b_v) > $3t$. Adopt 300 mm deep and 200 m broad
		A_s required for tension = $\frac{270 \times 1000}{180} = 1500$ mm^2
Fig. Ex. 11.1		Effective width acting = $\frac{b_0}{2} + \frac{1}{3}(b_0 + 16t)$ (part of shell acts with beam.)
		Assume depth, (d_0) = 300 mm
		Breadth (b_0) = 200 mm
		Eff. width = $100 + \frac{1}{3}(200 + 16 \times 70) = 540$ mm
		Length of slab in tension = 540 − 200 = 340 mm
		Total area = (300 × 200) + (340 × 70) = 83800 mm^2

Reference	Step	Calculations
		Tension in concrete = $\left(\dfrac{T}{A_s + mA_s}\right) \approx 2.5\ \text{N/mm}^2$
		(We may reduce the size of edge member by prestressing it.)
	4.	*Design of inner edge beam*
		(These are in compression and borders two shells)
		The inner edge beam will be in compression from two shells. The compression varies along the length.
		Slanting length = 7/cos 11.3° = 7.14 m
		Max. compression = 2 × 7.14 × 38.5 = 550 kN
		Assume allowable compression as that allowed by IS 456 but not more than 1/4 critical buckling value.
		We may proceed in two ways:
		Method 1: Check whether effective slab width of 16t (8t on either side) will be enough to take this compression. Then put the necessary steel.
		Method 2: Thicken the slab at junction as shown in Fig. Ex. 11.1 and find the steel required.
		As the compression is variable and member is also restrained by the slab, even though buckling is most unlikely, we will check it as follows.
	5.	*Check edge members in compression*
		Allowable stress to be less than 1/4 buckling stress.
Chapter 19		Reference [1] gives the following formula for critical load of umbrella roof. Assume M25 concrete.
Sec. 19.4		$q_{cr} = \dfrac{2Et^2h^2}{a^2b^2\sqrt{3}}$ and $\sigma_{cr} = \dfrac{q_{cr} \times a \times b}{2th} = \dfrac{Eth}{ab\sqrt{3}}$
		where t = thickness, h = rise, and a and b are sides.
		$\sigma_{cr} = \dfrac{25000 \times 70 \times 1400}{7000 \times 7000 \times \sqrt{3}} = 28.87\ \text{N/mm}^2$
		With F.S. = 4; σ_{cr} = 7.2 N/mm², i.e. $\sigma_c \not> 7.2\ \text{N/mm}^2$
IS 456		For elastic design σ_c = 6 N/mm² for M25 concrete. Hence adopt σ_c = 6 N/mm².
	5.	*Column design*
		When the load is applied on full surface, the load is central. It is customary to design the column for full dead load + eccentricity due to only half loading of shell under live load. We must also take into account the effect of the wind load that can act on the structure as a whole.

EXAMPLE 11.2 [Design of hipped hypar shell roof on four supports]

Design a hipped hypar shell roof on four supports to cover an area 20 m × 15 m as shown in Figure 11.6 and Figure Ex. 11.2.

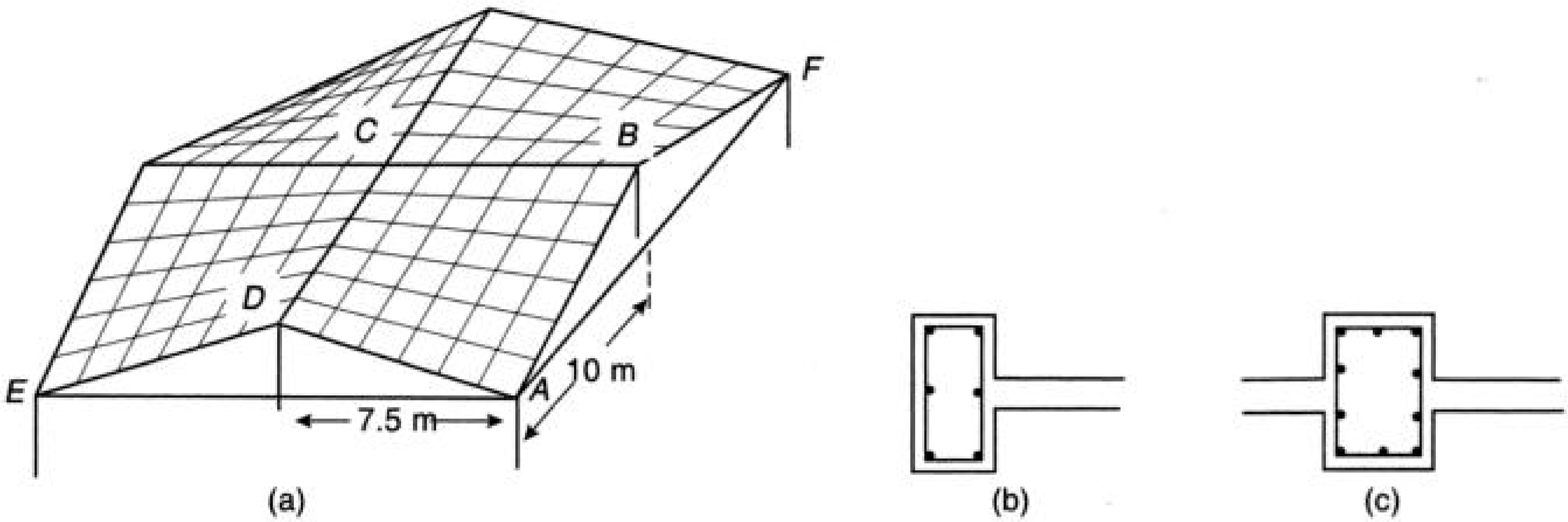


Figure E11.2 Details of a hipped hypar shell: (a) Layout, (b) Edge members, and (c) Ridge member. (See also Figure 11.6.)

Preliminary planning

Adopt four H.P. shells each of a = 10 m, b = 7.5 m joined together by top ridge beams and sloping edge beams as shown in Fig. Ex. 11.2. Ties are also to be provided along the sides between the corners.

$$a = 10 \text{ m};\ b = 7.5 \text{ m};\ t = 75 \text{ mm; Choose, } h = \frac{b}{5} = \frac{7.5}{5} = 1.5 \text{ m}$$

Loads, $\text{DL} = 0.075 \times 24 = 1.8 \text{ kN/m}^2$

Insulation = 0.2 kN/m^2

Live load = 0.5 kN/m^2

Total = 2.5 kN/m^2

[**Note:** We will indicate only the analysis of forces. Details of the design are given in Section 11.8.1.]

Reference	*Step*	*Calculations*
	1.	*Determine shear in the shell part*
Fig. 11.6		There are four hypar shells with straight edges each similar to *ABCD*.
		Shear forces in the shell 75 mm thick
		$= \dfrac{wab}{2h} = \dfrac{2.50 \times 10 \times 7.5}{2 \times 1.5} = 62.5 \text{ kN/m}$
Eq. (11.2)		Stress = $\dfrac{62.5 \times 1000}{1000 \times 75} = 0.83 \text{ N/mm}^2$ (This will produce equal tension in the shell.)
		(As tension is greater than 0.5 N/mm^2 and even otherwise, we provide full steel for tension.)
		$A_s = \dfrac{62.5 \times 1000}{140} = 446 \text{ mm}^2\text{/m 8 mm @ 100 mm} = 503 \text{ mm}^2)$

Reference	Step	Calculations
		We provide this along the sides both ways. In addition as there is bending of shell near the edge beam, provide extra 8 mm bars at 20 mm on top for a length of 1m from edges.
		We may also check for buckling as in Example 11.1.
	2.	*Design of sloping edge member AD in compression on 20 m side from D to A* (a = 10 m)
		Edge beams can be formed by thickening the shells as shown in Figure 11.7 or by separate beams as shown in Figures 11.6 and Ex. 11.2.
		Length = $\sqrt{(10)^2 + (1.5)^2}$ = 10.1 m
		Compression = 62.5 × 10.1 = 632 kN
		As compression varies and as member is also constrained, slenderness need not be considered.
		For compression member, provide 1% steel. Let section be A_c. Assume f_c = 4 N/mm².
		$$4 \times A_c + 130\left(\frac{A_c}{100}\right) = 63200$$ $$A_c = 12300 \text{ mm}^2$$
		Depth should be > 4t, say, 500 mm
		Breadth should be > 3t, say, 250 mm
		Area is > 12300 mm²
		Steel area = $\frac{500 \times 250}{100}$ = 1250 mm²
		Provide 6 nos. 16 mm bars (1407 mm²)
	3.	*Design of edge member AB on* b = 7.5 m
		Inclined length = 7.514 m
		Compression = 7.514 × 62.5 = 470 kN
		Design as in step 2.
	4.	*Ridge members BC and CD (This member is also in compression varying from zero at gable end to maximum at the centre of the shell.)*
Fig. 11.6		This beam collects the shear from the shells on both sides.
		Assume section 500 mm depth × 400 mm width
		(a) Ridge beam along (10 m + 10 m) side beam BC
		Total compression = 2 × 10 × 62.5 = 1250 kN
		(b) Ridge beam along (7.5 m + 7.5 m) side beam DC
		Total compression = 2 × 7.5 × 62.5 = 937 kN
		Design the beams for these values of compression.

Reference	*Step*	*Calculations*
	5.	*Horizontal tie member AE*
		This is an important member and should not be forgotten.
Fig. 11.6		Tension force is the horizontal component (Compression × cos β)
		$$T = \frac{(62.5 \times 10.1) \times 10}{10.1} = 625 \text{ kN}$$
		This should be designed for its self weight also. It is an ideal member to be prestressed to carry the dead weight of the beam and the tension. It is easy to prestress them also.
	6.	*Design for weight of edge and ridge members*
		As pointed out in Section 11.8, in calculations we have not taken account of the weight of the edge members. The weights are not carried by the shells. Hence, it is necessary to design the ridge beam members to take their own weight and also design the gables to support the beams.
	7.	*Detailing*
		H.P. is detailed according to the basic principles of detailing R.C. beams and columns (see Figures 11.6 and 11.7).

EXAMPLE 11.3 [Oblique hypar shells—Analysis of a hexagonal hypar shell roof shown in Figure 11.8]

Determine the stresses for design of a hypar shell on a regular hexagonal plan on a circular ring of 5 m in radius. The hypar consists of three oblique hypar shells as shown in Fig. Ex. 11.3, where the three inner members are horizontal and the six outer members

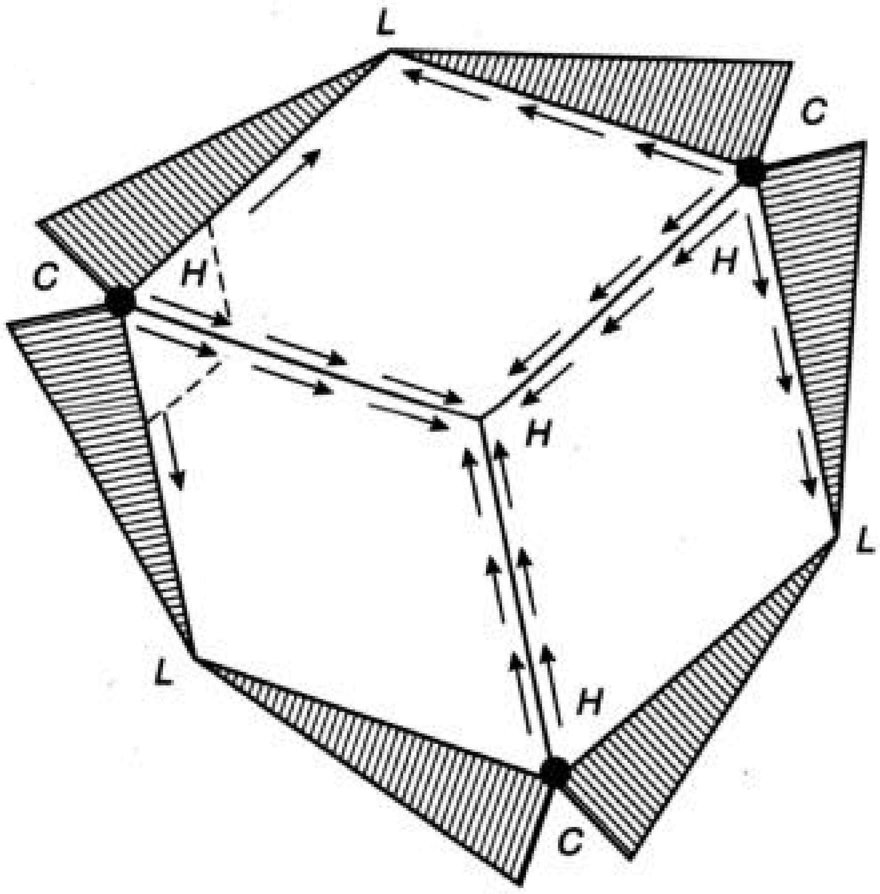


Figure E11.3 Oblique hypar on regular hexagonal base on three columns (C denotes position of columns.) H = High level, L = Low level. (See also Figure 11.8.)

are inclined downwards and supported at one end and unsupported at the other end. Assume the total loading of the shell is $w = 2$ kN/m^2 and the thickness of shell is 60 mm [1]. (Full theory is given in Reference [1].)

Reference	*Step*	*Calculations*
	1.	*Adopt a suitable rise (h) for hypar*
		Assume shallow shell $h = \frac{1}{5}$ (side)
		Length of all sides = a = 5 m
Sec. 11.5		Assume shallow shell $h = \frac{5}{5} = 1$ m
		$R_1 = \frac{2a^2}{h}\cos^2 60 = \frac{2\times 25}{1.0}\times\frac{1}{4} = 12.5\text{ m}$
		$R_2 = -\frac{2a^2}{h}\sin^2 60 = -37.5\text{ m}$
		Slenderness = $\frac{60}{12500} = \frac{1}{208}$
		[See Eq. (11.3)]
	2.	*Calculate shear force on the shell*
Ref. 1		$S = \frac{wab}{2h}\sin\alpha; \left[\alpha = \frac{360}{3} = 120°\right]$
		$S = \frac{2\times 25}{2\times 1}\times 0.866 = 21.65\text{ kN/m}$
	3.	*Calculate tension in the peripheral member*
		Max. tension = $S \times a = 21.65 \times 5 = 108.25$ kN
	4.	*Compression in the inner member*
		These are stressed by the shear from either side of the shell + the component of the tension from the two inclined members at the column junction in its direction. (The other component will balance itself at column.)
		$C = 2 \times a \times s + 2 \times (108.25) \times \cos 60°$
		$= (2 \times 5 \times 21.65) + 108.25 = 324.75$ kN
	5.	*Load on each column*
		As there are 3 columns and 3 paraboloids, each column takes load on one paraboloid.
		P = Load on one shell = $a^2 \sin 60 \times w$
		$= 25 \times 0.866 \times 2 = 43.3$ kN
		(Refer also to Example 2.3 in Reference [2].)

REVIEW QUESTIONS

1. What are paraboloids? What are the three types of popular paraboloids and how are they formed? How are they named?
2. Explain how the hyperbolic paraboloid can be generated as a ruled surface. What is meant by oblique hypar shells?
3. Name two types of hyperbolic paraboloids that are commonly used as roofs. How do you derive a hypar which is a rectangular ruled surface from one of them?
4. Sketch the common types of hyperbolic paraboloid roofs with straight edges.
5. What are the component elements of (a) an umbrella type hypar shell and (b) a hipped hypar shell?
6. Explain how the forces in the main hypar shells with straight edges can be imagined to be resisted by arches. Explain how this concept can be used to determine the stresses in the shell.
7. How do we design the edge members of a hypar shell?
8. What are oblique hyperbolic paraboloids? Sketch two types of roofs using oblique hypar shells. Sketch a hyperbolic paraboloid shell that can be built on a hexagonal plan. What are the forces acting on each of its members?

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12

DESIGN OF PARABOLIC CONOIDS

12.1 INTRODUCTION

Conoid is a singly ruled surface formed when a straight line generator moves along a curve C_1 at one end and a straight line at the other end as shown in Figure 12.1. The last two (curve and straight line) are called the *directrices of the conoid*. The planes of the generator and the directrices are assumed to be at right angles to each other. Conoids have negative Gaussian curvature and hence are anticlastic surfaces.

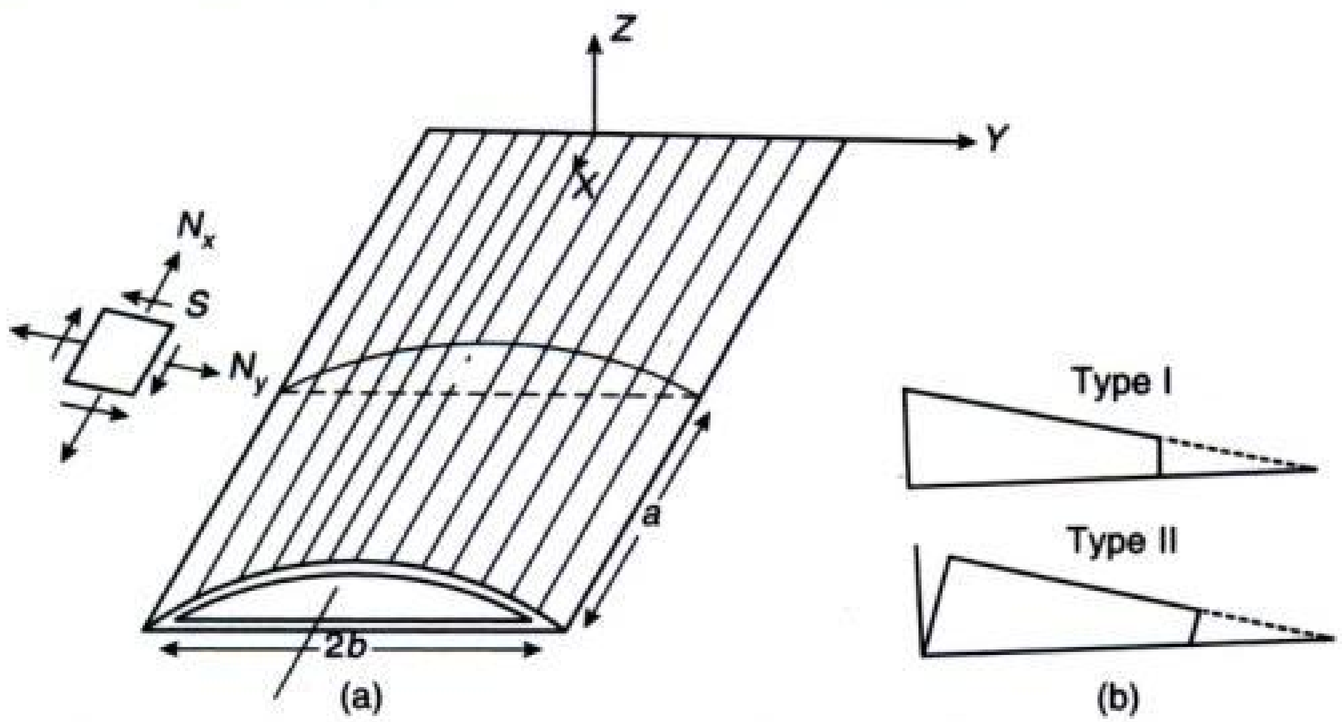

Figure 12.1 Formation of a parabolic conoid: (a) Singly ruled surface formed by a straight line moving over a parabola at one end and a straight line at the other end and (b) Type I and Type II cenoids.

Conoids are frequently used as northlight shells for factories (Figure 12.2). They can give considerable amount of natural lighting. Also as a singly ruled surface, it is easier to make formwork for its construction than for cylindrical northlight roofs.

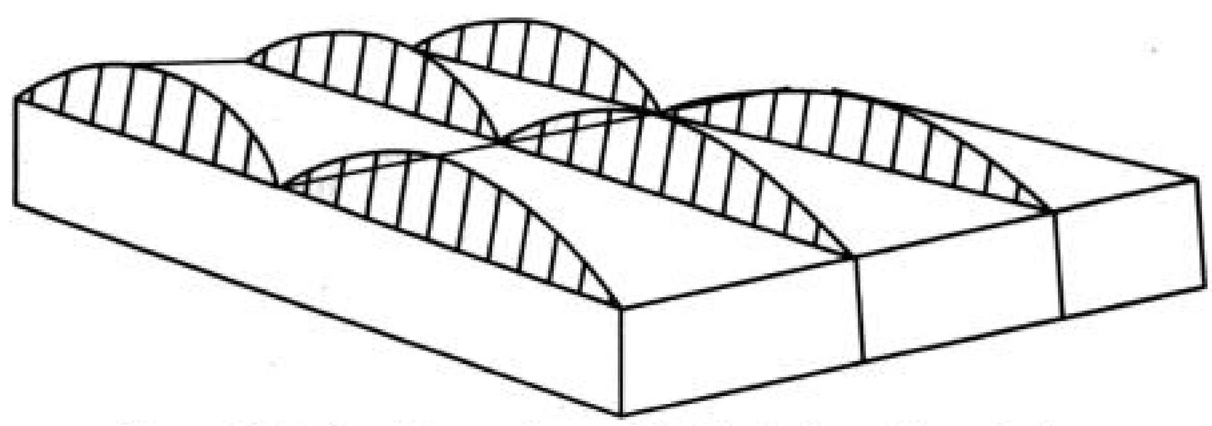

Figure 12.2 Conoids used as northlight shell roof for a factory.

Conoids with *vertical northlight* openings are called Type I conoids. Conoids with *inclined northlight ends* which can allow more lighting are called Type II conoids (Figure 12.1). In this chapter, we deal only with Type I conoids. (see Section 12.7 for Type II conoids.)

A conoid is named after the directrix curve C_1. This can be a circle, parabola, ellipse, catenary, etc. resulting in circular, parabolic ellipsoid or catenary conoids, respectively. In this chapter, we study only *parabolic conoids* which are the most commonly used type of conoid. Generally, they are truncated as shown in Figure 12.1 so that their real span a is less than the theoretical span L. The span of the parabola is taken as the breadth and the truncated distance as the span of the conoid. *Generally, the breadths of conoids are larger than their spans.* Some recommend the following breadth to span ratios:

(a) Long conoids whose *breadth is only* $1\frac{1}{3}$ *times* the span (spans up to 18 m are in common use)
(b) Short conoids whose *breadth is as much as three times* the span (spans up to 8.5 m of this type are in common use)

[**Note:** Published data and tables giving results of complete analysis of parabolic conoids for design for these types of shells are available in Reference [1]. It is better to use either these tables or modern computer programs for actual analysis of these shells than compute the stresses from basics. In this chapter, we adopt the same convention as used in the chapters on cylindrical shells, tension as positive and compression as negative. This is used in C and CA publication also.]

12.2 PARTS OF A CONOID

A parabolic conoid, as in the case of cylindrical shell, can be divided into the following three parts, as shown in Figure 12.1, for design:

- The shell proper
- *The edge beams on the sides*
- The parabolic transverse (also known as diaphragms) on *each end*.

The symbols used to describe the dimensions are the following (see also Figure 12.5):

Span or length = a Thickness = h

Breadth = $B = 2b$ Edge beam = $b_2 \times h_2$

12.2.1 Edge Beams

Edge beams can be supported in two ways: (a) Type are those supported on closely spaced supports (columns) so that beams are free from bending moment and deflection and (b) Type B are those supported on elastic beams the beams being supported only at their ends on the end transverses (See Figure 12.4). Details of these edge beams are given in Section 12.6.3.

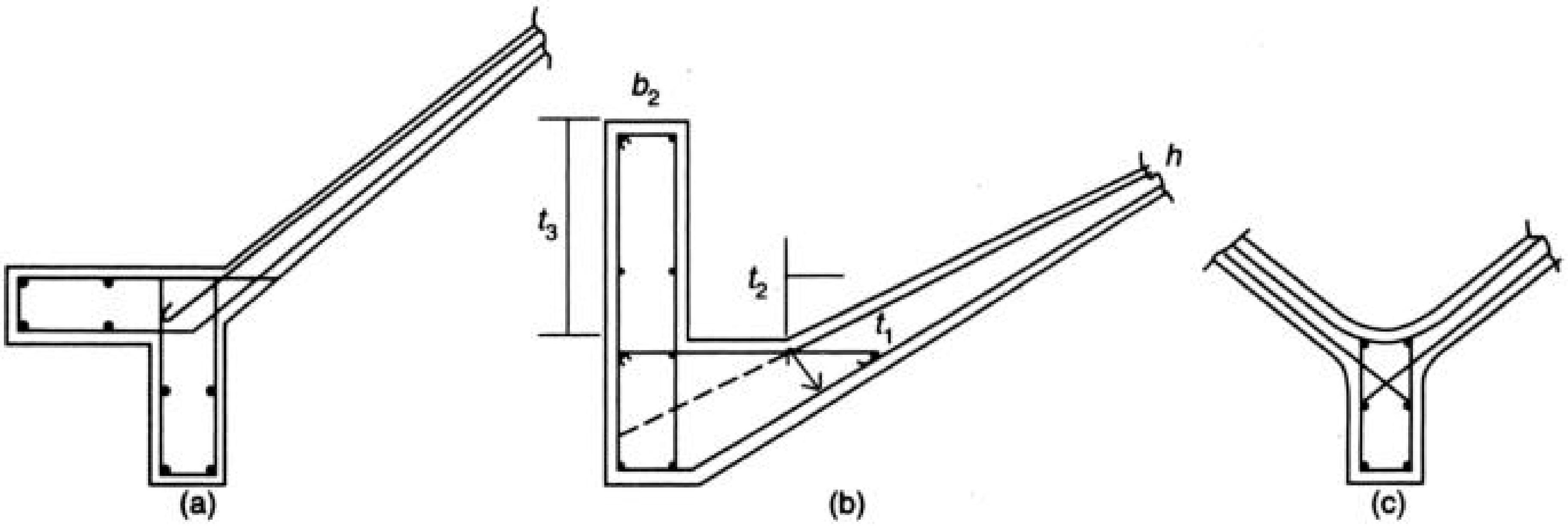

Figure 12.3 Types of edge beams used for conoids: (a) Commonly used for single shell. The horizontal part taken the horizontal component of edge force, (b) Edge beam recommended by C and CA (U.K.), and (c) Edge beam for multiple conoids.

12.2.2 End Supports for Conoids (Transverse)

Transverses are rigid supports placed at the ends of the shell at the high and low ends. The shells are thickened near these supports. These supports at the ends are usually pinned arches with a tie. When conoids are used as northlight shells as shown in Figure 12.2, the high end is usually supported by an upper arch and the low end can be supported on a lower arch or the tie member as shown in Figure 12.4. (Arches can span larger length than beams). For a 75 mm thick shell, we may adopt an arch of size 300 mm deep and 200 mm width for a conoid of width 12 m.

12.2.3 Edge Thickening

The shells themselves are made thicker at the edge beam side and also near the transverse arch side as follows to merge with them.

At the side edge beam side, the details of thickening can be as shown in Figure 12.3(*b*) [5]. In design, the additional weights are added to the weight of beams due to thickening.

h = thickness of main parts of shelll = 60–70 mm

t_1 = thickness with junction with edge beam = $a/120$ but not less than 125 mm and not greater than 230 mm

t_2 = breadth of gutter = 250 mm

t_3 = height above gutter = 600 mm

b_2 = breadth of edge beam = 230 mm

(Total depth of edge beam will be of the order of 1000 mm)

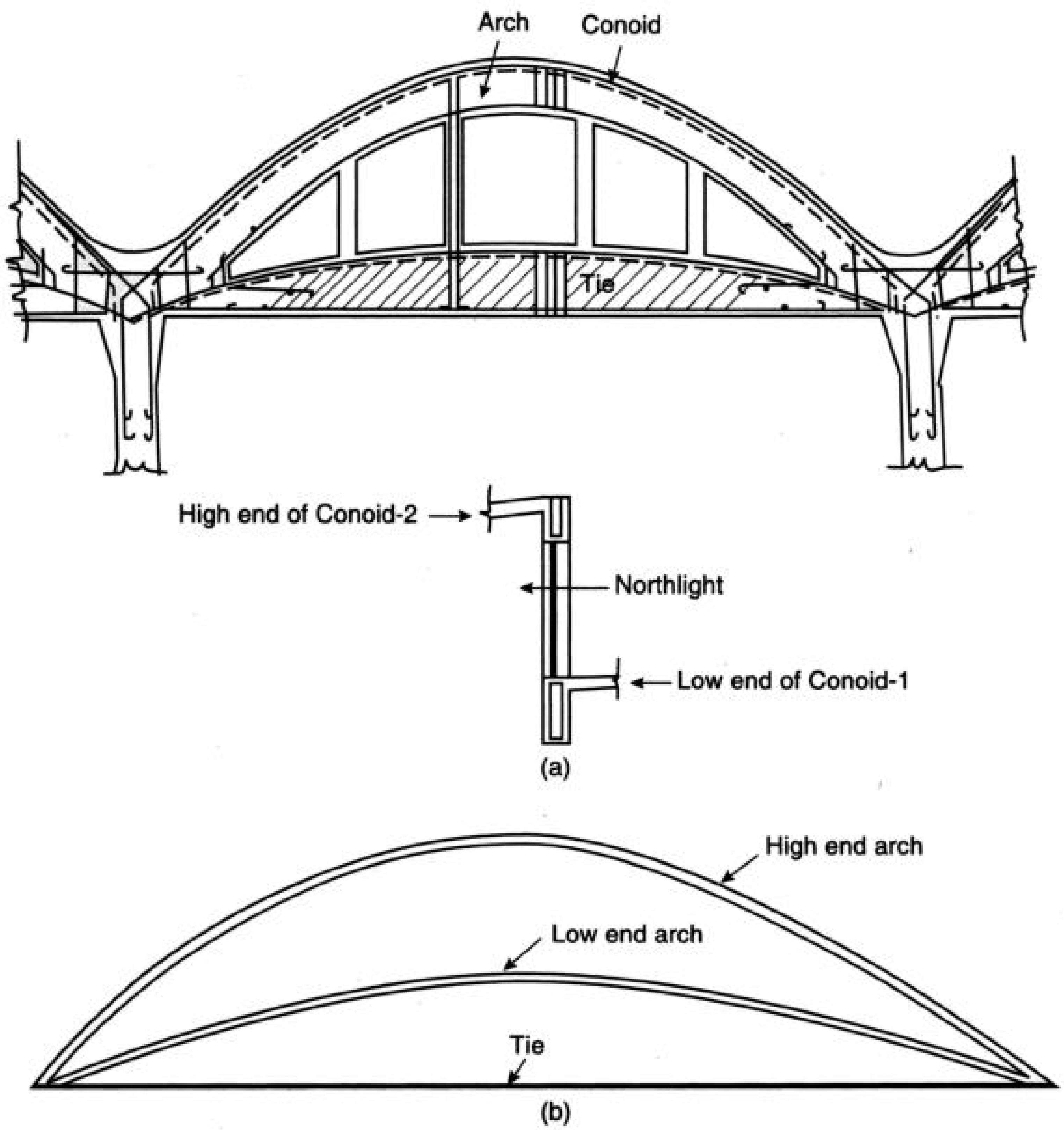

Figure 12.4 Transverse in conoids: (a) High end on arch and low end on the tie, and (b) High and low ends on arches with additional tie.

The thickening is carried out adjacent to the end transverses also. It is usually accomplished by thickening the shell ends by an additional 25 mm for a distance equal to 1/10th the span from the transverse. The transition should be smooth and gradual. Extra reinforcement should be also provided in these thickened part to take care of edge bending.

In the C and CA (U.K.) publication on conoids [1] giving values of the various forces for design, conoids are divided into the following two groups:

1. Conoids with $a = 3/4\ B$, which we will call *long conoids*.
2. Conoids with $a = 1/3\ B$, which we will call *short conoids*.

[**Note:** In general in conoids value of a will be less than B.]

12.3 NATURE OF STRESS RESULTANTS IN THE SHELL PART OF CONOIDS

As in other types of shells, we can have *membrane analysis* to determine the stresses N_x, N_y and N_{xy} in the main parts of the shell. However, the boundary discontinuities produce moments also in the shell boundaries. In many cases, as conoids are difficult to analyze, we compute only the membrane forces and then detail the shell boundaries by thickening and putting extra steel so that the moments are taken care of by themselves. An exact analysis needs bending analysis of the shell. However, the membrane analysis of conoids itself is quite involved and the bending analysis is more complicated. Hence, published data giving tables of values for design (as that in Reference [1]) or computer software output are more often made use for bending analysis. As shown in Figure 12.1, the membrane forces are N_x in X-direction, N_y in Y-direction (**considered positive in tension and negative in compression**) and N_{xy} forces (**considered positive, increasing in Y-direction**). Bending analysis gives M_x, M_y and M_{xy}. However, it can be proved that the major stress resultants in the shell part of the conoids to be taken are only N_x, N_y, N_{xy} and M_y, four in number. Force N_{xy} is the shear which can also produce diagonal tension and compression. M_y is the moment in the transverse direction. **Bending moment in shells is considered positive when it is a sagging moment producing tension in the underside of the shell**, and negative when it is hogging producing tension on top of the shell. Figure Example 12.1 gives us an indication of the nature of variation of the membrane forces in conoids. A comparative study with cylindrical shell is interesting.

12.4 GEOMETRY OF PARABOLIC CONOIDS

Most books on shells take the coordinates as shown in Figure 12.5 and Figure 12.6, with the origin on the low end on the straight line end. This is referred as the "commonly used" or "conventional system". Others, as in the publication by Cement and Concrete Association of U.K. (Reference [1]) discussed in Section 12.5, take the origin at the high end as shown in Figure 12.6.

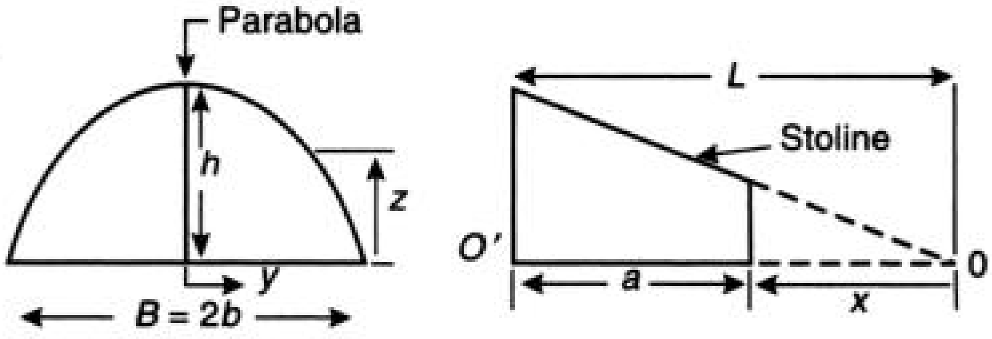

Figure 12.5 Geometry of parabolic conoids (see also Figure 12.1).

12.4.1 Equation to Shell Surface with Conventional Origin [Figure 12.5]

The equation to the shell surface with reference to the conventional coordinates can be described as follows with reference to Figure 12.5. (The coordinates for C and CA publication are shown in Figure 12.5).

As the directrix is a quadratic parabola with rise f, the *equation to the parabola* will be,

$$z_1 = f - f\frac{y^2}{\left(\frac{B}{2}\right)^2} = f\left[1 - y^2\left(\frac{4}{B^2}\right)\right] \tag{12.1}$$

$$= \frac{4f}{B^2}\left[\frac{B^2}{4} - y^2\right]$$

z_1 is the ordinate at the high end.

As the surface is formed by a straight line moving over the parabola and the straight line through the origin, z at x from the origin (with the distance of the high end from the origin being equal to L as shown in Figure 12.1 taking z-axis downwards) will be,

$$z = -z_1\frac{x}{L} = -\frac{4f}{B^2}\left(\frac{B^2}{4} - y^2\right)\left(\frac{x}{L}\right) \tag{12.2}$$

Thus, with the origin on the straight line at the low end and z-axis pointing downwards, the equation for a parabolic conoid can be expressed as follows:

$$z = \left(\frac{4f}{B^2}\right)\left(\frac{B^2}{4} - y^2\right)\left(\frac{x}{L}\right)$$

where

f = Rise of the central part of the high end

$B = 2b$ = Full breadth (b = half breadth)

L = Theoretical span of conoid (The real span of the conoid shell will be always smaller than the theoretical span as only a truncated conoid is used for the purpose of roofing.)

12.4.2 Equation to Shell Surface with C and CA Coordinates

The C and CA publication takes the origin under the high end as shown in Figure 12.5. Using the same symbols as used in the C and CA publication, Let

Span = a

Width = B and half width of conoid = $b = \dfrac{B}{2}$

Let

H = Height at the high end

H_0 = Height at the low end

$$g = \left(1 - \frac{H_0}{H}\right)$$

Then equation to the parabolic conoid surface with origin O' below the centre of the high end, x-axis along length a, y-axis along the breadth b and z-axis downwards as shown in Fig. 12.5 is,

$$z = -H\left(1 - \frac{gx}{a}\right)\left(1 - \frac{y^2}{b^2}\right) \tag{12.3}$$

12.4.3 Shallow and Deep Conoids

As we saw in the case of hyperbolic paraboloids, when we load paraboloid shells, for simplification we may assume that the loads are uniformly distributed over the horizontal projection of the shell. This is true only if the shell is shallow. In deep shells analysis, we have to divide the load into two parts and solve for the membrane stresses. We first find the projected stress resultants and then the actual stress resultants. Parabolic conoids are considered as shallow when

$$\tan^2 \psi = \frac{(16f^2)}{B^2} \tag{12.3a}$$

is less than unity [1]. If it is larger than unity, it is considered as a deep shell.

12.5 MEMBRANE STRESSES

As in other forms of shells, the three important membrane (tangential) stresses are N_x, N_y and N_{xy}. In addition, for good design we have the bending forces M_x and M_y, of which the more important is M_y. For the analysis of membrane forces for conoids, *instead of using three separate equations of equilibrium, we use Airy type* of stress function (given by Puncher) by which the three equations reduce to a single second order partial differential equation in terms of the stress function and this is solved for the membrane forces [3], [4].

[As already stated for the general solution of a deep conoid, we split the load into two parts, of which one load is applicable to shallow conoids where the load can be considered as uniform on the plan of the shell.] Thus, we have the general analysis which yields the values of $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ on the projected plan area and the real values of the stresses N_x, N_y and N_{xy} are calculated from them. We will not go into the mathematical derivation but only point out that detailed derivations can be obtained from References [2], [3], [4], [5].

12.5.1 Approximate Formulae for Membrane Stresses

References [3] and [4] give the following formulae for approximate membrane stresses in parabolic conoids with *conventional coordinates* (origin O at the end of the straight line as shown in Fig. 12.5). First, the projected values on plan area $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ and then the real values N_x, N_y and N_{xy} can be expressed as follows (For details, see Reference [3]): Let

L = Length of the large end from origin

x_0 = Length of small end

a = Span of the conoid = $(L - x_0)$

B = Breadth of conoid = $2b$

w = Unit load on the shell

$$k = \frac{8f}{LB^2}$$

The membrane stresses at x, y from the origin are

$$\bar{N}_x = -wk\left[\frac{3}{16}(L-x)y^2 - \frac{B^2}{32}(L-x) + \frac{1}{6}(L^3 - x^3)\right] \tag{12.4}$$

$$\bar{N}_y = \frac{wk}{2}(xy^2) - \frac{w}{kx}\left[1 + \frac{1}{2}\left(\frac{f}{L}\right)^2\right] \tag{12.5}$$

$$\bar{N}_{xy} = -wk\left(\frac{y^3}{16} - \frac{yB^2}{32} + \frac{x^2y}{2}\right) \tag{12.6}$$

[Note: $\bar{N}_x = 0$ at $x = L$ and $\bar{N}_{xy} = 0$ at $y = 0$.]

For a shallow shell, we can adopt these as the N_x, N_y and N_{xy} values. In general, to convert them to N_x, N_y and N_{xy}, we use the following relations:

$$\text{Taking } \frac{\partial z}{\partial x} = p \quad \text{and} \quad \frac{\partial z}{\partial x} = q$$

$$\cos\phi = \frac{1}{\sqrt{1+p^2}} \quad \text{and} \quad \cos\psi = \frac{1}{1+q^2} \tag{12.7}$$

$$N_x = \bar{N}_x\sqrt{\frac{1+p^2}{1+q^2}} \tag{12.8}$$

$$N_y = \bar{N}_y\sqrt{\frac{1+q^2}{1+p^2}} \tag{12.9}$$

$$N_{xy} = \bar{N}_{xy} \tag{12.10}$$

[Notes:

1. In deep shells, generally there will be a reduction of stress by this operation.
2. The maximum slope at high end = $\tan^{-1}\left(\frac{2f}{b}\right)$.
3. We may imagine that if we extend the shell to the straight line, the part near the straight line will be a slab or plate with a lot of bending. Hence, we truncate it considerably to reduce this bending effect.]

12.6 DESIGN TABLES FOR CONOIDS PUBLISHED BY CEMENT AND CONCRETE ASSOCIATION (UK) [5]

The determination of even membrane stresses in conoids by calculations (as given in References [2] to [6]) is difficult. Hence, the Cement and Concrete Association (C and CA)

of UK has published Design Tables for a series of shallow shells with fixed dimenstions [1]. By using the general theory of shells due to Marguerre (in which the membrane theory is only a special case of the general theory), values for moments M_x, M_y and M_{xy} as well as the tangential forces N_x, N_y and N_{xy} are presented for a number of shells of fixed dimensions.

In this publication of C and CA, the surface load for a 65 mm thick shell is assumed 0035 N/mm^2 (3.5 kN/m^2) as shown in the example given in the publication. **Tension is taken as +ve and compression as –ve. Sagging moments with tension at lower part of the shell is taken as positive**. *The forces are in* N/mm *and moments given in* N.mm/mm. The tables are presented with origin below the high end. The Y-axis points along the half breadth (b) horizontally, the X-axis points along the length (a) towards the low end, and Z is taken vertically downwards. The following cases are dealt with in this publication:

Case (1): The edges are supported by closely spaced columns so that the stiffening beams carry only axial loads.

Case (2): The edges are supported on beams whose torsional and horizontal flexural rigidities are negligible.

The shells considered can be classified into two groups: (a) where the breadth B is only 4/3 span, which we will call *long shells* and (b) where the breadth B is three times the span, which we will call *short shells*. Fixed dimensions of edge beams are also specified.

12.6.1 Dimensions of Conoids Analyzed in C and CA Publication

We must note that the C and CA publication has a geometry with its origin different from what we find in the conventional books. The conoid is laid out as shown in Figure 12.5. *The origin is under the high end*, the X-axis is towards the low end, and the Y-axis is along the breadth.

a = Length or span of the shell

$B = 2b$ = Breadth of the shell

H = Height of high end

H_0 = Height of low end

As the tables published by C and CA (UK) are "flexural theory design tables", they can be considered more accurate than what we can get by formulae for membrane analysis. It is advisable to use these tables or available special computer software based on bending theory for all practical designs.

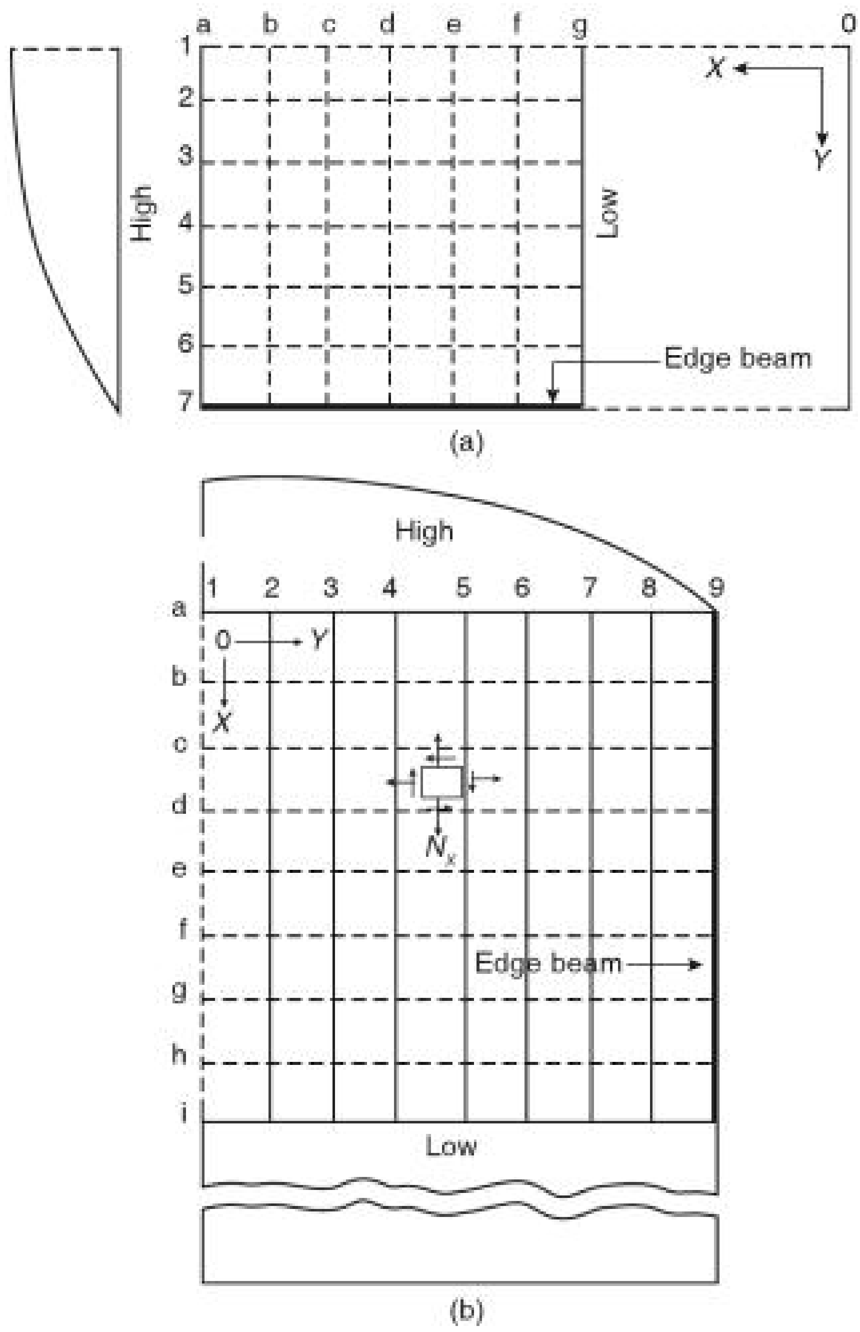

Figure 12.6 Coordinate systems: (a) Conventional system with origin on straight line and (b) Coordinate system used in C and CA (UK) manual (Ref. [1]).

TABLE 12.1 Sample of Design Table Published by C and CA, UK [1]

(M_y, N_x, N_y and N_{xy} for conoid with closely supported edge beams) [See Example 12.4]

a = 15000; $2b$ = 20000; H = 3750; H_0 = 1875; h = 65; b_2 = 230; h_2 = 1000; w = 3.5 kN/m^2

[Weight of shell = 0.065 × 25 = 1.62 kN/m + other loads is taken as 3.5 kN/m^2 as shown in the example].

As $a = \frac{3}{4}B$, it is to be considered as a long shell. From page 31 of the publication, we get the following values. [+ve values of N_x and N_y is tension–M_y considered positive when it produces tension in the inner fibres.]

(Horizontal line corresponds to the nine points a to i of Figure 12.6.) The thick line represents the edge beams.)

M_y (Nmm/mm)

High End

1	2	3	4	5	6	7	8	9
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
–316.4	–228.5	–224.9	–86.0	250.6	487.0	869.9	86.7	–2464.9
–565.7	–476.0	–422.5	–125.8	501.1	1053.3	1395.9	–93.0	–3767.0
–734.6	–663.3	–558.9	–128.0	704.8	1471.0	1653.0	–294.9	–4361.5
–840.7	–722.1	–639.7	–168.0	802.4	1592.5	1944.4	–212.0	–5269.1
–774.0	–673.6	–624.4	–213.3	699.0	1544.4	2000.9	–158.2	–5307.1
–627.5	–486.6	–534.0	–278.2	483.5	1179.9	2023.7	150.0	–5484.8
–443.0	–209.4	–357.2	–275.3	276.2	520.9	1708.2	557.3	–5057.0
0.0	0.0	0.0	0.0	–0.0	–0.0	–0.0	–0.0	0.0

Low End

N_x (N/mm)

High End

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.6	9.4	–11.9	–32.1	–31.5	–56.9	–17.0	42.9	23.6
12.5	12.5	–19.8	–54.8	–69.2	–84.4	–28.4	45.4	50.1
22.6	16.2	–24.7	–71.0	–94.4	–102.2	–41.0	41.1	71.2
32.3	18.9	–23.5	–74.6	–108.6	–111.6	–62.2	18.0	91.3
37.3	20.1	–17.0	–66.1	–108.7	–112.2	–81.5	–8.8	102.1
38.1	18.4	–6.0	–45.2	–93.3	–98.2	–100.2	–47.6	104.6
32.7	9.2	3.4	–16.3	–62.3	–54.1	–87.8	–80.1	83.4
–0.0	–0.0	0.0	0.0	0.0	0.0	0.0	0.0	–0.0

Low End

N_y (N/mm)

High End

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
–55.4	–53.5	–53.9	–52.7	–47.9	–44.0	–23.7	0.3	0.0
–55.6	–56.6	–57.2	–57.2	–54.6	–45.0	–31.6	–16.9	0.0
–54.7	–54.7	–55.4	–53.3	–45.3	–32.0	–14.1	–1.2	0.0
–69.1	–70.3	–72.5	–72.7	–68.0	–56.8	–40.7	–21.0	0.0
–64.2	–65.3	–67.5	–67.9	–62.9	–50.5	–32.2	–19.9	0.0
–79.3	–80.3	–85.0	–90.1	–91.4	–86.1	–65.3	–31.5	0.0
–87.4	–93.3	–101.5	–116.6	–137.3	–146.0	–146.5	–103.7	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Low End

N_{xy} (N/mm)

High End

0.0	–2.0	–4.9	15.1	34.9	68.5	108.0	86.3	49.8
0.0	–4.8	–4.1	8.9	31.2	60.0	82.3	79.4	60.6
0.0	–5.8	–4.0	4.0	23.3	42.7	51.3	57.6	51.4
0.0	–4.8	–5.3	–1.2	9.3	20.7	31.6	43.9	41.6
0.0	–3.9	–4.8	–8.9	–6.5	–2.5	3.8	26.4	34.0
0.0	–0.5	–3.3	–11.5	–19.7	–25.0	–20.9	–3.3	6.0
0.0	1.4	1.2	–14.8	–30.8	–48.9	–58.9	–30.2	–6.7
0.0	10.1	12.6	3.8	–20.7	–55.8	–93.1	–120.0	–108.2
0.0	18.6	20.2	24.3	–6.7	–51.1	–100.4	–193.8	–207.3

Low End

[**Notes:** C and CA publication gives values of M_x, M_y, M_{xy}, N_x, N_y, N_{xy}, V_x, V_y and deflection values. Only the important stresses M_y, N_x, N_y and N_{xy} are given above. The values of N_x and N_y at high and low ends are zero.]

[Assumed load = 3.5 kN/m^2 space; thickness of shell h = 65 mm. Values are in N and mm units as indicated in the above tables. (See 4.2, Example of that publication.)]

The equation for the shell surface is Eq. (12.3). The thickness of the shell is h = 65 mm. The edge beams are as shown in Figure 12.3(b) with given depth h_2 and thickness b_2 as shown in Table 12.1. A weight equal to 1.5 times the weight of the edge beam is assumed as the weight of the shell part that acts with the beam.

The following two sets of conoids have been analyzed in Reference 1.

Set 1: Long shells: $a = \frac{3}{4}B$ with fixed H and H_0 values as $H = \frac{a}{4} = \frac{3}{16}B$; $H_0 = \frac{H}{2}$

B in metres – 8, 9, 10 … 25 (18 nos.) (long shells)

Set 2: Short shells: $a = \frac{1}{3}B$ with fixed H and H_0 values as $H = \frac{a}{2} = \frac{B}{6}$; $H_0 = \frac{H}{2}$

B in metres – 18, 18.75, 19.75 … 24.75 (10 shells in increments of 0.75 m) (Short shells)
(Both these sets, as defined by section 12.2.1, are shallow shells.)

Two sets of tables, one for the edge supported at close intervals (Type A) and the other on edge beams (Type B), have been presented. Table 12.1 gives the results of a shell closely supported along its edge (of set 1) B = 20 m, $a = 20 \times \frac{3}{4} = 15$ m. Table 12.2 gives the results of the same shell with the edge beam supported at its ends only and then allowed to bend.

The values will be different from that obtained from Eq. (12.5) to (12.7). We note $N_x = 0$ at both high and low ends.

Example 12.4 is a worked out example using Table 12.1.

[**Note:** We must be aware the values in the C and CA Tables are as represented is for a shell with the origin at the high end and with the y-axis along the breadth of the shell and the x-axis along the span of the shell. The values near the edge beams are represented by the values along the vertical line at the far right of the table. The layout of the table corresponds to Figure 12.6.]

12.7 ANALYSIS OF TYPE II CONOIDS

As we have seen, Type II conoids have inclined northlight surfaces. Type II conoids can be solved by rotation of the vertical loads. Reference 4 deals with the subject in greater detail.

12.8 DESIGN OF VARIOUS ELEMENTS

The design of the shell proper, the diaphragm and the edge beams can be summarized in the following manner. An examination of the stress resultants as given in Examples 15.1 shows the distribution of the *membrane stress resultants*. The reinforcement layout is shown in Figure E12.2. Details are as follows.

12.8.1 Design of Shell Proper

The stresses in the shell are generally small as shown in Examples 12.1 and 12.3. In practice, we assume that only N_x, N_y, N_{xy} and M_y are the important forces to be considered for design.

We generally provide two layers of steel, one on the top and the other at the bottom, as grids so that the percentage of steel is not less than minimum of 8 mm at 200 mm as specified in IS 2210 (1988). The spacing is to be less than or 3 times the effective depth. We generally proceed as follows:

1. *General reinforcement:* As the minimum size of steel to be provided is 8 mm, we may try a grid (on the top and bottom of the shell) spaced at 200 mm (for a shell 100 mm thick). There will be bottom and top layers (i.e. two layers of steel) in the body of the shell.
2. *Providing for N_y:* The N_y forces are mostly compression and the thickness of the shell will be ample. In addition, we have the nominal steel to provide small tensions that may occur in some areas towards the low end.
3. *Providing for N_x:* The N_x forces can be tension in most areas. It is especially large at the high end. The nominal steel provided will take most of the force. It is critical near the edge beam at high end and the central part of the shell. If necessary, some extra steel can be provided in that area in X-direction.
4. *Provide for N_{xy}:* These shears are most critical near the edge beam sides at the high end. Special mesh or inclined steel can be provided for these shears as shown in Figure Example 12.1.

5. *Providing for bending moment M_y in the shell:* Even though membrane theory does not give us the bending moment, the value of M_y is important and is present because of the edge disturbance produced by the edge beam. If we use the membrane theory, we can only thicken the shell near the edge beam side for a distance of not less than 1/10 span and not more than 1/6 span. The slab is thickened to span/120 and not less than 125 mm or not more than 230 mm (section 12.2.3). One of the arrangements suggested for thickening near the edge beam and providing extra steel for M_y is shown in Figure 12.4 and section 12.2.3. The moment M_y is usually negative (hogging) near the edge beam producing tension on top for a distance not less than 0.15 span.
6. If we get the value of M_y from bending theory, we can use the value for the design of the thickness and the steel required.

As already stated, the general arrangement of steel is first to provide two layers of nominal steel, one at the top and the other at the bottom of the shell proper. Further, in places where extra steel is required for N_x, N_y or M_y, these are introduced. Extra steel is also provided for N_{xy} (shear steel) as in the case of cylindrical shells at the corners of the high end or as a steel in both directions. The publication by C and CA (Reference [1]) gives detailed guidelines for its design.

12.8.2 Design of End Transverses (Diaphragms)

As the width B values of conoids are large, we adopt arches for the diaphragms (Figures 12.1 and 12.4). Their design is similar to that we dealt with for the design of transverses in cylindrical shells. The reaction between the shell and the diaphragms is caused by the N_{xy} (shear forces) in the shell near the arch. These produce compression and tension. For analysis, the arch is usually divided into eight equal parts along its length (four on either side of centre as shown in Figure 12.7) and the vertical and horizontal forces due to the shear are determined. The arch is usually designed as a tied two hinged arch.

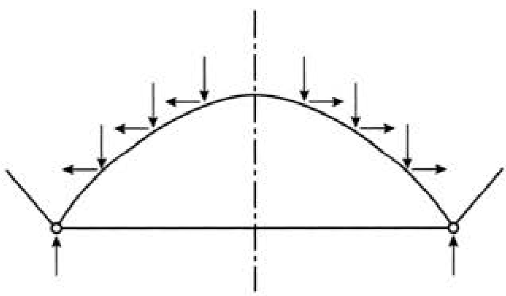

Figure 12.7 Design of transverse (arch support).

The lower part of the diaphragm can either be a closed tie member or another arch at the low end below the upper arch as shown in Figure 12.4. The low end is designed as a tie of the arch. As the tie is in tension, we may use higher strength concrete, say, M40 for the tie only and can be prestressed.

In the arch analysis, finding the horizontal thrust is a standard problem in theory of structures. The standard practice is to divide the arch into 8 equal parts of length ΔX, four on each side of the centre line. The coordinates of the centre of these divisions can be determined. The magnitude of V and H acting on these points of the arch are,

$$V = N_{xy} \frac{\Delta X}{\cos\theta} \sin\theta = N_{xy} \cdot \Delta X \cdot \tan\theta$$

$$H = N_{xy} \frac{\Delta X}{\sin\theta} \cos\theta = N_{xy} \cdot \Delta X \cdot \cot\theta$$

The shear from the shell can be assumed to act on the shell tangentially. The (average shear force in each division × length of the division) can be assumed to act at the centre of each division. These forces can be resolved into vertical and horizontal components as shown in Fig. 12.7 to design the arch[5].

In addition, we have also to take the weight of the arch. The horizontal thrust in the arch can be found by unit load method.

[Note: Details of calculation of arches can be obtained from Reference [5]. Because of the large breadth of these shells, it is best to support these shells on arches. In general, we use arches to reduce the bending moment in long span structures as horizontal tensions produced in the tie member considerably reduce the bending moment in the main arch. For conoids, we adopt a two hinged arch system with hinges at the base, which is statically indeterminate only to the first degree. The theory of design of these arches is available in Reference [4] and other books on theory of structures.

We can also note, by applying the theory of design of the diaphragm for cylindrical shells that the point of zero shear along the length of the conoid is *usually* away from the high side so that more of the load is transferred on the high end than the low end. This is also evident from the distribution of N_{xy} around the upper and lower sides of the conoid.]

12.8.3 Design of Side Edge Beams

It is very important to design side edge beams correctly. Though for exact analysis we may use the method of design of edge beams of cylindrical shells, we commonly use the following approximate method for its design:

Method 1: C and CA publication divides edge beams into two types, those supported by closely spaced columns (Type 1) or wall (on craft paper) and those supported only at the ends so that they can bend freely (Type 2).

(a) Type 1 edge beams

In Type 1 beams, the only effect is the tension produced by N_{xy}, the shear acting at the junction with the shell. This total tension is given by,

$$\int_0^x N_{yx} dx \text{ at the junction}$$

We can plot the value of shear N_{yx} along X-axis.

12.9 DETAILING OF REINFORCEMENT

The general arrangement of steel in parabolic conoids is as shown in Figure 12.8. Reference [5] may be consulted for more details.

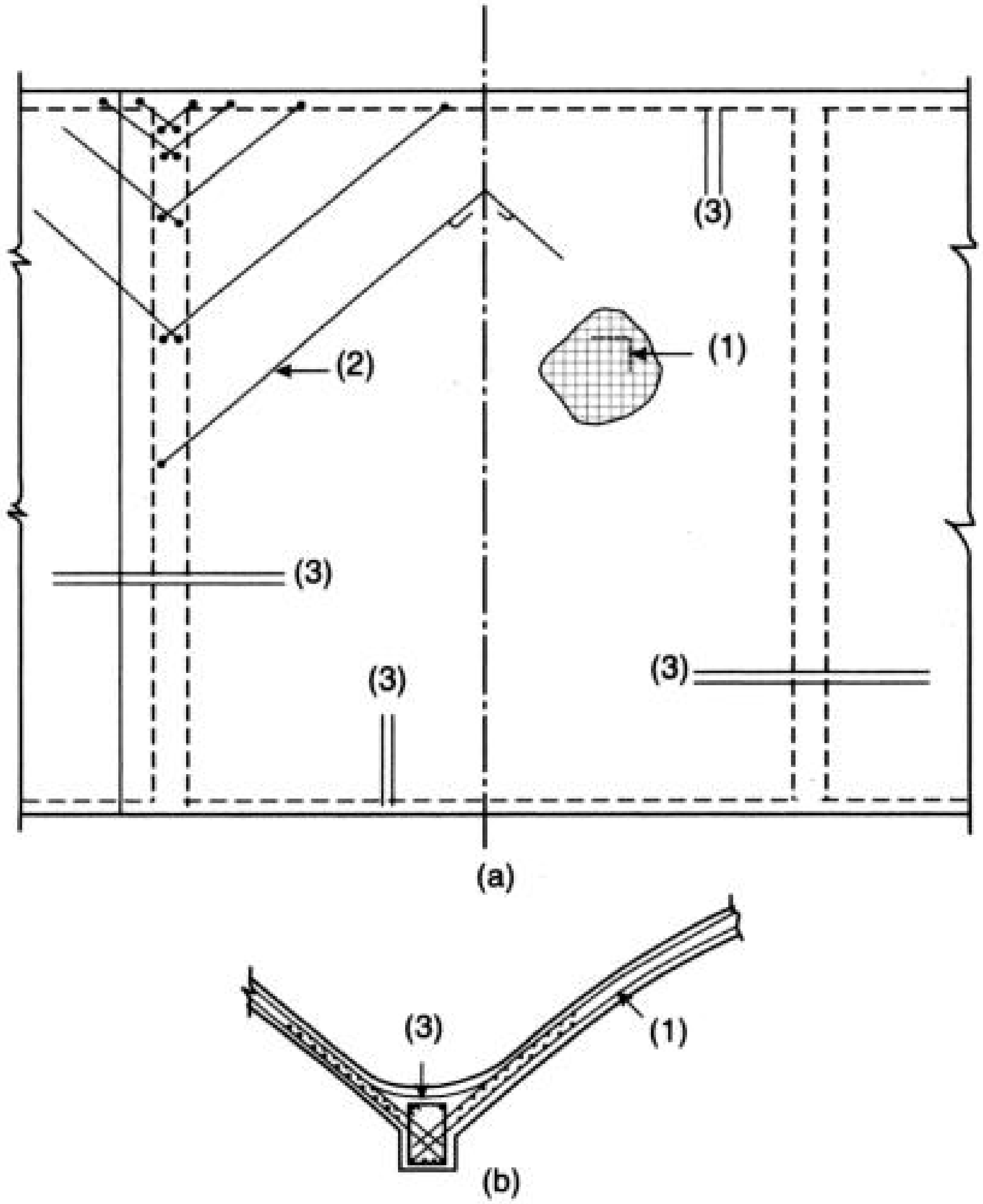

Figure 12.8 Detailing of steel reinforcements in conoids: (a) Plan and (b) Edge beam of multiple conoids. (1) Steel placed at top and bottom of shell both ways, (2) Diagonal steel, and (3) Extra steel placed at thickened edges.

SUMMARY

This chapter dealt with membrane analysis and design of conoids. The C and CA publication gives the results of total (bending and membrane) analysis of a number of shells of fixed dimensions. Formulae for membrane analysis have been presented. The design of the end supporting diaphragms and the edge beams has also been presented in this chapter.

EXAMPLE 12.1 (Calculation of membrane force in a parabolic conoid [1] [2])
(The following is a worked out example given in References [2] and [3]. As already stated, the calculation of even the membrane forces in conoids is very tedious and not trustworthy. Hence, the results of the worked out example in the above reference are presented below *to explain the nature of forces acting on parabolic conoids* and the design of the conoid.)

Dimensions of a conoid

Theoretical span (L) = 12 m

Real span (a) = 8 m

Width (B) = $2b$ = 20 m (Note $B > a$); b =10 m

Rise (H) = 4.5 m

Shell thickness = 100 mm

Total load = 3 kN/m^2

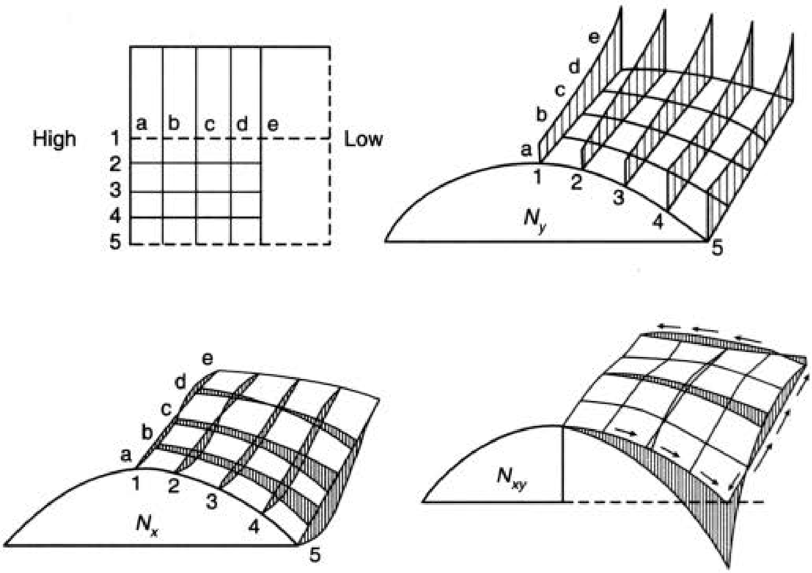

Figure E12.1 Nature of distribution of membrane forces in parabolic conoids (compare with those in circular cylindrical shell. Note N_{xy} in conoids is large at high ends.)

Calculation (Refer Figure 12.6)

Check whether shell is shallow (Section 12.4.1) Eq. (12.3a)

$$\frac{16H^2}{B^2} = 16 \times \frac{(4.5)^2}{(20)^2} = 0.81 < 1.$$

Hence shallow.

We take the conventional coordinates with origin on straight line as shown in Figure Ex. 12.1. Divide the span a = 8 m into four equal divisions, each of 2 m and name these ordinates along a as a, b, c, d and e (do not get point 'a' confused with a) as shown in Figure Example 12.1. We get ψ values as follows:

Point a = $\dfrac{x \text{ from origin}}{L} = \dfrac{12}{12} = 1$ (Origin is taken on straight line)

Point b = $\dfrac{10}{12}$ = 0.833; Point c = $\dfrac{8}{12}$ = 0.67

TABLE 2 Ex 12.1 N_x(kN/m) distribution

← X; ↓ Y; High end (left); Low end (right)

	η \ ψ	a 1.0	b 0.833	c 0.666	d 0.5	e 0.333
1	0.0	0	0	0	0	0
2	0.25	0.94	–0.10	–0.29	–0.07	0.28
3	0.50	0.25	–0.12	–0.68	–0.27	0.43
4	0.75	5.4	–0.05	–1.27	–0.72	0.32
5	1.00	10.3	0.17	–2.17	–1.57	–0.20

As already stated, this example has been incorporated to *study the nature of distribution* of N_y, N_x and N_{xy}. It is shown by Figure Ex. 12.1.

The distribution can be noted as follows:

1. N_y. The major stress resultant N_y is compressive (–ve). Taking points along Y-direction starting from the axis, its value increases towards the edge beam (unlike in cylindrical shells). Along X-direction, it increases towards the low end.
2. N_x. The value of N_x is mostly in tension. It is high near the high end, the larger tension being near the edge beam side.
3. N_{xy}. The value of shear N_{xy} is maximum at the high end near the edge beam and reduces to zero along the central axis. The direction of N_{xy} at this high end is such as to produce tension cracks along lines joining the high end *to the junction of transverse and edge beam*. (Hence shear steel has to be placed normal to these cracking as shown in Figure Ex. 12.1(b). This is similar to the shear steel in cylindrical shells.)

[**Note:** However, we must be aware that the above results are from membrane analysis. Bending analysis will introduce bending M_x, M_y and M_{xy}, of which M_y will be the most important. The results of bending analysis of conoids with given dimensions can be obtained from the publication by Cement and Concrete Association, London, which is explained in Example 12.3.]

We will see that M_y produces maximum negative (hogging) moment near the longitudinal edges. It sharply reduces and, at about one quarter span distance, reverses in sign to sagging moment. We must check the top and bottom layers of steel to take care of these hogging and sagging moments.

Detailing of the conoid is as shown in Figure Ex. 12.1.

EXAMPLE 12.2 (Analysis of side edge beam and parabolic transverse of Example 12.1) [2] Indicate how to arrive at the loads on the edge beam for its *approximate design* as given in Reference [2].

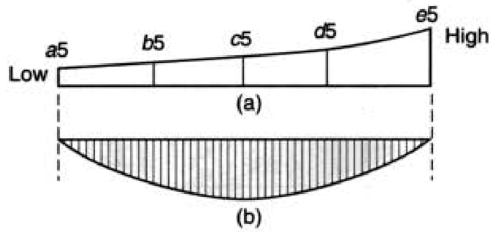

Figure E12.2 Design of edge beam: (a) Loading on edge beam is due to dead load of beam + live load on beam + vertical component of N_y on beam. (Horizontal component to be taken separately), and (b) Bending moment diagram due to vertical loads.

Reference	*Step*	*Calculations*
		Part 1—Design of edge beam
		[**Note:** Strictly speaking, we should design the edge beam as in cylindrical shells obeying compatibility of forces and deformation. However, this is very cumbersome for conoids. Hence, each method tries to satisfy equilibrium of forces only.]
		The following is a summary of the method followed in Reference [2].
Sec. 15.7.3	1.	*Adopt layout for the edge beam*
		We adopt a combined vertical and horizontal edge beam as shown in Figure 12.4(a). We divide the edge beam into four parts a_5, b_5, c_5, d_5, e_5 along the length. The vertical loads on the edge beam are taken as the weight of the beam and the vertical component of N_y. The horizontal load is taken as the horizontal component of N_y. The angle at which N_y acts can be found for the point from the equation to the shell surface.
		Let weight of the edge beam and LL = 5 kN/m
		Usually, the depth of beam is taken as 1/15th span and the width of the beam not less than 230 mm for normal conditions.
	2.	*Calculate the vertical and horizontal components of N_y at a_5 to e_5*
		$V = N_y \sin \beta$ and $H = N_y \cos \beta$ [Figure 12.3(a)]

Point	N_y(kN/m)	*b*	(kN/m)	
			V	*H*
a_5	86.2	42°	57.7	64.1
b_5	64.5	37°	38.7	51.6
c_5	62.5	31°	32.1	53.6
d_5	74.7	24°	30.4	68.2
e_5	108.1	16.7°	31.1	103.6

Reference	Step	Calculations

3. *Design the vertical part of edge beam* (Δl = 2 m)

	a_5	b_5	c_5	d_5	e_5	
DL + LL =	5.0	5.0	5.0	5.0	5.0	(kN/m)
N_y load =	57.7	38.7	31.1	30.4	31.1	(kN/m)
	62.7	43.7	37.1	35.4	36.1	(kN/m)

We may approximate the bending moment as $WL/8$. Use Simpson's rule to find total load from the ordinates = (Total vertical load) × (Span)/8.

4. *Design the horizontal part assuming as beam supported between tiers*

Find BM due to horizontal load. The horizontal depth resisting its effect in design is taken as that of the horizontal beam plus 1/6th part of the span of the shell which is thickened. (Ref: Section 12.8.3)

4(a) [**Note:** Another way of design of edge is described in the Cement and Concrete Association pamphlet [1]]. The following is a summary of this method.]

[Edge beam design as given in Reference [1] is as follows.

In Reference [1], edge beams are divided into (a) Type 1 – those closely supported along its length and (b) Type 2 – those supported at their ends only.

In Type 1, the force acting on the edge beam is taken as the tension due to N_{xy} at the junction. It can be found by averaging the tensions as shown in Table 2 of Example 12.4.

In Type 2 edge beams, the forces are N_y (assumed small) and N_{xy} tension as in Type (1) and self weight of the beam. For details, refer Reference [2].]

Part II—Design of parabolic end diaphragm (transverse)

1. *Layout of the transverse diaphragm*

It is laid out as a two pinned arch as described in Section 12.8.2.

2. *Determine the forces for design*

The arch is divided into eight parts, four on each side of the centre line, and the vertical and horizontal forces due to N_{xy} in the shell and the weight of the arch rib are taken as loads. With a series of conoid shells one behind the other, the lower end of the conoid in front also ends as a lower arch or on the tie member [Section 12.8.2].

EXAMPLE 12.3 [Planning a conoid shell roof (a) as long shell and (b) as short shell as specified in C and CA, UK publication, Reference [3]]

Give the layout of (a) long conoid and (b) short conoid for a span of 6 m as given in C and CA design tables.

[**Note:** The data given in the tables are for fixed dimensions.]

Reference	*Step*	*Calculations*
Sec. 12.6.1	1.	**Case (1) Long conoid** ($B = 2b = 4/3a$) **or 1.33*a***
		Estimate the dimensions for long conoid
		Span length a = 6000 mm = 6 m
		Breadth B = 4/3 span = 8000 mm
		Thickness = 60 mm
		Front rise H = 1/4 span = 1500 mm [It can be 1/3 to 1/4 span.]
		Rear rise $H_0 = 1/2H$ = 750
		Depth of edge beam = 1/15 span = 400 mm
		Breadth = 230 mm
	2.	**Case (2) Short conoid shell** ($B = 2b = 3a$)
		Span length a = 600 mm = 6 m
		Breadth $B = 3a$ = 18000 mm = 18 m
		Thickness h = 60 mm (say)
		Front rise H = 1/2 span = 3000
		Rear rise $H_0 = 1/2H$ = 1500
		Depth of edge beam = Span/15 = 6000/15 = 400 mm
		Thickness of edge beam = 230 mm
		[**Note:** Between the two types, only B differs. The same size of edge beam is generally used for both.]

EXAMPLE 12.4 [Analysis of a long conoid a = 15, $2b = B$ = 20 m with longitudinal edges closely supported, using tables published by C and CA, U.K., Page 31]

Shell dimension actual span a = 15 m

Breadth $B = 2b = \dfrac{4}{3} \times 15 = 20$ m (long conoid)

High rise $H = \dfrac{1}{4}$ span = 3.75 m

Low rise $H_0 = \dfrac{1}{2}H$ = 1.875 m. Hence $(L - a)$ = 15 m

Depth of edge beam = $\dfrac{1}{15}$ span = 1000 mm

Thickness h = 65 mm; breadth of beam = 230 mm

Assumed surface load (as specified for the given table) = 3.5 kN/m^2 (see Tables from page 31 of Reference [1]).

The values of N_y, N_x, N_{xy} and M_y are tabulated in grid form in Tables 1 to 4 on page 172 to 175 as given in the C and CA publication.

Reference	*Step*	*Calculations*
		[**Note:** Tables 1 to 4 given at the end of this example give values of N_y, N_{xy}, N_x and M_y reproduced from the C and CA tables, pages 31.] After getting these forces, we proceed as follows for the design of the shell.]
	1.	*Design shell for compression and tension*
		(1) Allowable compression (σ_{ce} = 5 N/mm^2)
		= 5 × Thickness × 1 mm = 5 × 65 = 325 N/mm
		All values of N_y given in Tables 1 and 3 of Ex. 12.4 below are below this value.
		(2) Allowable tension
		Minimum steel = 0.12 × Gross area
		$= \dfrac{0.12 \times 65 \times 1000}{100} = 78 \text{ mm}^2/\text{m}$
		(IS 2210 (1988) Clause 12.2.5 recommends minimum diameter of steel as 6 mm and minimum spacing 200 mm both ways top and bottom.)
		Max. spacing = 3*d* or 300 mm
		Min. diameter of roof steel = 8 mm
		Assume a layer of 8 mm @ 200 mm mesh provided at the top and bottom (2 layers)
		Steel area for 8 mm @ 300 mm = 167 mm^2/m
		Both top and bottom = 334 mm^2/m
		Allowable tension $= \dfrac{334 \times 230}{1000} = 76.8$ N/mm
		In the tables, +ve values are tension and –ve values are compression.
		The N_x and N_y forces are compressive and hence safe. Shear produces tension. Tension is exceeded in N_{xy} table (Table 2). It requires checking for shear steel.
Fig. 12.3	3.	*Bending moment (M_y)*
		We have large BM near the edge beam. We can thicken the edges and provide extra steel. Thickness of shell = 60 mm
		Thicken shell near edge beam $t_1 = \dfrac{\text{Span}}{120} \nless 125$ mm and $\ngtr 230$ mm
		$= \dfrac{15 \times 1000}{120} = 125$ mm. Adopt 150 mm
		Smooth thickening length $= \dfrac{\text{Span}}{6} = \dfrac{15}{6} = 2.5$ m. We also check for thickness and steel for the moment M_y.

Reference	*Step*	*Calculations*
	4.	*Design for maximum M_y of Table 4, Ex. 12.4*
		M_y max. = 5485 N.mm/mm (shell closely supported)
		Using Fe 415 steel and elastic design
		$M = 0.131\ \sigma_b bd^2$
		With M25 concrete, σ_b = 8.5 (allowable)
		$M = 1.11\ bd^2$
		$d = \sqrt{\dfrac{5485}{1.11 \times 1}} = 70.33$ mm
		Hence thicken 65 mm to d = 135 mm
		$A_s = \dfrac{5485 \times 1000}{230 \times 0.9 \times 135} = 194\ \text{mm}^2/\text{m}$
		Provide 10 mm at 300 which gives 264 mm²/m

TABLE 1, EX. 12.4 Values of N_y(kN/m, i.e. N/mm (From pages 31 and 61 of C and CA publication [1]) (Refer also to N_y values in Table 12.1)

[**Note:** X-axis is along the span and Y-axis along the breadth. Upper values are for conoids with beams closely supported (same as Table 12.1 of this text) and table on page 31 of C and CA publication). The lower values are for beams supported at their ends from page 61 of C and CA publication.] (Points 1, 2, 3, 4, 5 correspond to points 1, 3, 5, 7 and 9 of the publication. Similarly a, b, c, d, e represent alternate points).

Origin → Y; ↓ X

	1	2	3	4		5 (Edge Beam)
a	0	0	0	0	0	0
b	−55	−57	−54	−31	−16	0
	−63	−56	−48	−27	−11	
c	−69	−72	−68	−40	−21	0
	−70	−74	−67	−46	−32	
d	−79	−85	−91	−65	−31	0
	−73	−89	−30	−114	−90	0
	−87	−101	−137	−146	−103	0
	−66	−82	−100	−90	−48	0
e	0	0	0	0	0	0

Low End

Comments:

1. Only approximately half the number of (alternate) values given in the original text are recorded in this and subsequent tables.
2. N_y forces are mostly negative (compression). Assuming allowable compression of 5 N/mm for concrete.
3. Allowable compression with nominal steel = 5 × 65 (thick nos.) = 375 N/mm.

TABLE 2, EX.12.4 Values of N_{xy}(kN/m or N/mm) from pages 31 and 61 of C and CA Publication [1] (Refer also to N_{xy} values of Table 12.1 of this book)

[**Note:** Upper values are for case (1) conoids with beams closely supported and lower values are for case (2) edge beams supported at their ends.]

↓X / →Y	1	2	3		4			5
a	0	–4	34	10	8	86		149
		1	25	8	7	115		121
		–4	41	5	1	57		51
b	0	–4	19		8	64	2	74
			4					
	0	–4	–6		3	26	–7	34
c		–18	–10	2	0	32		26
	0	1	–36		–58	–30		–7
d		4	–19		–48	–64		–74
	0	12	–40		–93	–140		–108
		25	–9		–103	–137		–137
		20	–6		–100	–193		–207
e		36	–5		–127	–178		–173

Low End

Comments:

1. Maximum value at high end is 108 and at low end is 207 for case 1, Reference [1], page 79.
2. Assume allowable shear in concrete as 0.55 N/mm^2. Shear force allowed = 0.55 × 65 (thickness) = 35 N/mm. Larger values occur at high and low ends adjacent to the edge beam. We have to put extra steel. It is easier to put both longitudinal and transverse steel to take care of these forces. See Figure 12.8 for placing of shear steel.
3. Axial tension in the edge beam = $(T_1 + T_2)/2$ where T_1 and T_2 are tension at top and bottom. Design of side edge beam and end transverses are described in the text. (Effect of tension similar to that in edge beams of cylindrical shells.)

TABLE 3, EX. 12.4 Values of N_y(kN/m or N/mm) from pages 31 and 61 of C and CA Publication [1]
(Refer also to N_x values of Table 12.1 of this book)

[**Note:** Upper values are for case (1) conoids with beams closely supported and lower values are for case (2) edge beams supported at their ends.]

High End

$1 \longrightarrow Y$, $a \downarrow X$

	1	2	3		4			5
a	0	0	0		0			
b	12.5 / 6	–19 / –10	–69 / –60		–28 / –63	45 / –38		50 / 0
c	32 / 46	–23 / –12	–108 / –113	–6 / –10	2 / 8	–18 / 9	/ 0	91 / 36
d	38 / 56	–6 / –5	–93 / –90		–100 / –131	–47 / –52	0 /	104 / 50
	32 / 36	3 / –4	–62 / –45		–87 / –92	–80 / –29		83 / 87
e		0	0		0			0

Low End

Comments:

Tensile forces occur near the edge beams and central part of the shell. Assuming elastic design with steel stress of 140 N/mm^2, required steel for 104 N/mm^2 = 104/140 = 0.74 mm^2/mm or 740 mm^2/m. Maximum spacing of steel allowed is 5 times the thickness of the shell (3×65 = 215 mm). Adopt 2 layers of 8 mm at 125 mm giving 804 mm^2. At the centre, we require much less steel for N_x. (The minimum steel specified in shells by IS 2210 (1988) Clause 12.2.5 is 8 mm at 200 mm centres.)

TABLE 4, EX. 12.4 Values of M_y(kN.m/m or N.mm/mm) from pages 31 and 61 of C and CA Publication [1]
(Refer also to M_y values in Table 12.1 of this book)

[**Note:** Upper values are for case (1) conoids with beams closely supported and lower values are for case (2) edge beams supported at their ends.]

X ↓ / Y →	1	2	3	4		5	
a	0	0	0	0	0	0	
b	–565 / –1039	–422 / –133	501 / 881	1395 / 179	–93 / –243		–3767 / –1628
c	–840 / –135	–639 / –521	802 / 1125	1944 / 1407	–212 / 8	0 / 0	–527 / –4993
d	–627 / –806	–534 / –525	483 / –61	2023 / 3458	150 / 1416		–5484 / –9560
	–443 / –446	–357 / –313	276 / –211	1708 / 2543	557 / 1328		–5057 / –7150
e		0	0	0			0

Low End

Comments:

1. The upper values are for closely supported beams and lower values for beams supported at ends only.
2. Both sagging (+ve) and hogging (–ve) moments occur in the shell (see Section 12.3). Maximum values of M_y are at longitudinal edges and are –ve. We find steel by elastic design.

EXAMPLE 12.5 (Calculation of Membrane Stresses by Formulae)

Using Eq. (12.5), calculate the N_y values for a conoid at along the middle of the shell at the centre and at the edges of the shell with the following data [4]. [Here we use the conventional coordinates.]

L = Distance of high end from origin on the st. line = (40 ft) = 12.19 m [4]

B = (60 ft) = 18.29 m

a = (30 ft) = 9.15 m

H = (12 ft) = 3.66 m = (f)

w = (60 lts./sq. ft) = 2.94 kN/m^2

$$N_y = \frac{wk}{2}\left([xy^2] - \frac{w}{kx}\left[1+\frac{1}{2}\left(\frac{f}{L}\right)^2\right]\right) \qquad \text{[Eq. (12.5)]}$$

$$k = \frac{8f}{LB^2} = 7.18 \times 10^{-3}$$

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13

DESIGN OF GROINED SHELLS

13.1 INTRODUCTION

The word 'groin' in architecture refers to "edge formed by intersecting vaults or arches". In this chapter, we first consider domes with ribs or hips as shown in Figure 13.1.

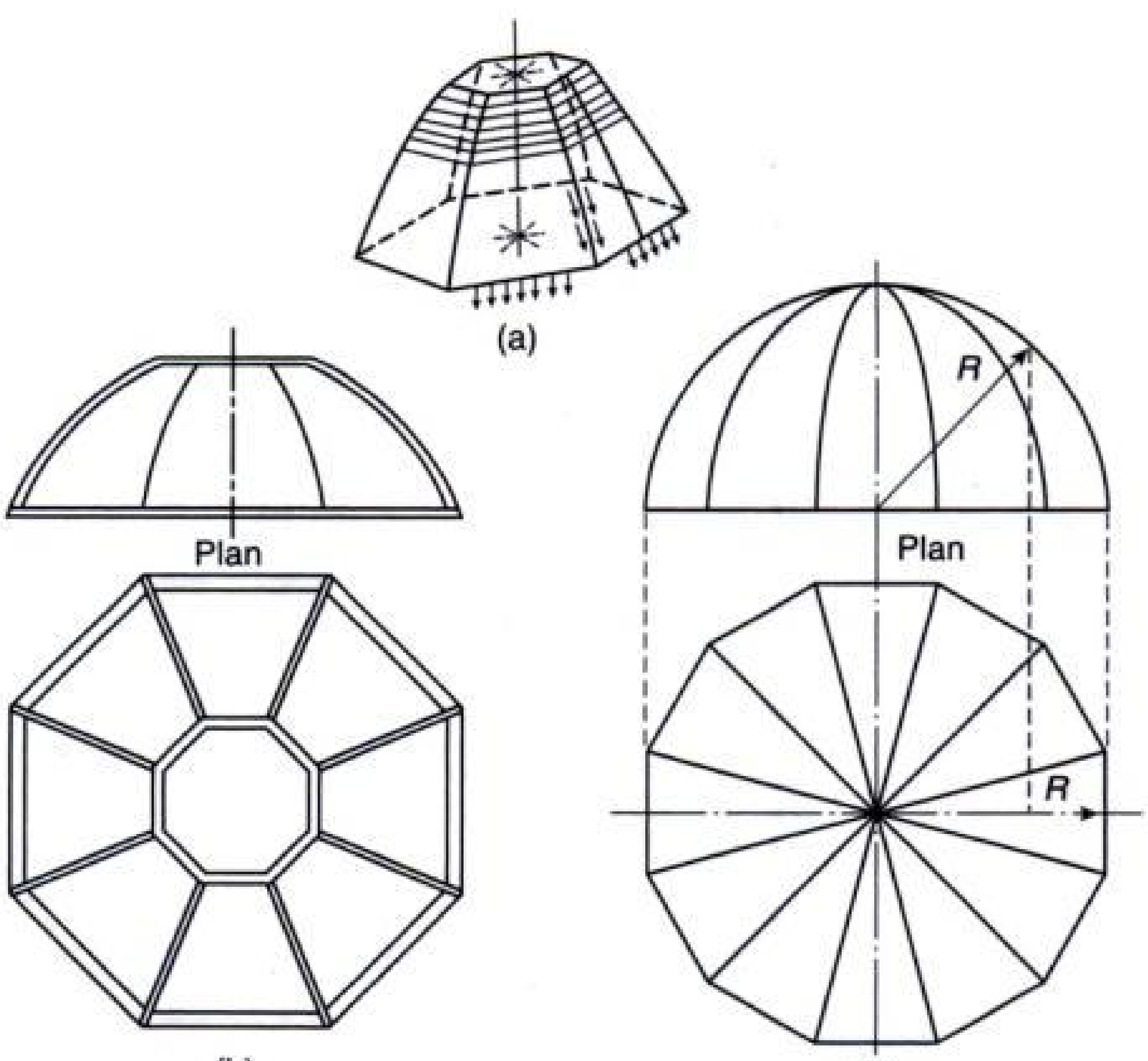

Figure 13.1 Polygonal domes with ribs: (Shells between ribs can be of many types, circular shell, parabolic shell or even a slab).

In Chapter 4, when we examined the action of reinforced concrete circular domes of uniform section, we found their limitations of producing hoop tensions. If we examine the architecture of old masonry domes, we can see that *ribs* were extensively used in them to get stability of brick domes [1] (see Appendix A). By use of ribs, the problem of tension in uniform circular domes can be overcome. We will examine some of the modern types of R.C. domes in which groins or hips are provided. We then examine the hyperbolic parabolic groined vaults in more detail.

13.2 POLYGONAL DOMES

Regular polygonal domes are those which use ribs as shown in Figure 13.1 [4][5]. The dome over the now famous Taj Hotel in Bombay built before the advent of reinforced concrete in India is of this type. In these shells, every horizontal section of the dome is a polygon, instead of a circle. The ribs of the shell pass through the corners of the polygon. The shell part is in between the groins. It has to have a *base ring* at the bottom, also connecting the groins and taking the reactions from the groins and from the shells. In addition, if it is open at the top, a lantern ring should be provided there. Thus, the important parts of polygonal domes are:

- Shell proper
- Groins (hip or rib)
- Base ring
- Top ring if there is an opening on the top.

The shell proper is in between the groins. It can be of many shapes, viz. cylindrical, parabolic, etc. The theory of polygonal domes is a good example of the role of ribs. As the ribs are subjected to balancing forces from each opposite side, we may assume that there are only axial forces along these ribs. The ribs are to be designed for their self-weight and the axial forces. References [3] and [4] give a good account of the analysis and details of design of these shells.

It can be easily seen that the incidence of tension that is present in uniform circular domes can be eliminated in these ribbed domes. As stated in Appendix A, these devices were used even by our ancient architects for stability of masonry domes.

13.3 FANLIGHT SHELLS

Another type of shells using ribs is the fanlight shells shown in Figure 13.2 for regular hexagonal and regular square plans [4]. These are groined shells, the intersection acting as an arch.

Here the sectors or component individual shells lie in a plane perpendicular to the side of the polygon. Each sector has a circumferential arch. The sectors can be hyperbolic cylindrical or any other type of shell. Their centre lines are perpendicular to the circumferential arch (arch along the circumferences). Here again, the design can be made

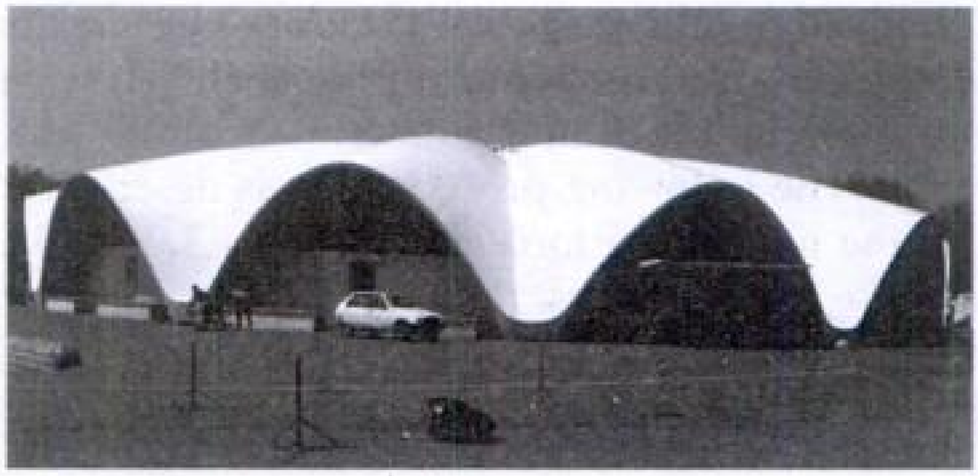

Figure 13.3(b) Example of a groined shell, Market Hall at Royan, France. The dome has a diameter of 55.5 m (185 ft).

The geometry of hyperbolic paraboloid surfaces is shown in Figure 13.3 [5]. These are made of two parabolas of opposite curvatures and hence they are anticlastic hyperbolic paraboloid shells. It is the same hypar we studied in Chapter 11, but with curved edges. The surface is formed by two parabolas, one with height h_x and base $2a$ and the other with height h_y and base $2b$. The equation for the surface can be written as

$$z = h_y\left(\frac{y}{b}\right)^2 - h_x\left(\frac{x}{a}\right)^2 \tag{13.1}$$

where h_x, h_y, a and b are as shown in Figure 13.3 (h_x and a along x-axis). Each term of the equation represents a parabola.

[If h_x (the small height) is made zero, then the surface is only a parabolic cylinder described in Section 13.5. It is easier to analyze these groined paraboloid shells with square plan $a = b$. Compare this equation with an elliptic paraboloid given in Section 11.10.]

As was done in Chapter 11 for hypar shells with straight edges conceptually, we may consider that the load transfer in these shells is carried out by a series of arches perpendicular and parallel to the free edges. As the arches normal to free outer edge are unrestrained and the circumferential arch at the edge can give only little support, we can assume that all the load is carried by the large arches parallel to the free edge. The load is thus carried directly to the groins which, in turn, act as arches. There are four separate HP shells in the total unit. Each is a replica of the other. The arches transfer all the load to the groins at the intersection. As the shell is symmetric with respect to the groins, the resultant load on the groin is a pure axial thrust.

The groins in turn can be assumed to act as two pinned or fixed arches according to the fixity at the ends. If the surface at the crown is quite solid as in large spans, the crown need not be considered as a hinge so that we will have a two hinged arch with hinges at supports. But for small span, because of lack of stiffness at the crown, it may be assumed as another hinge so that it will be a three hinged arch.

13.4.1 Calculation of Membrane Forces in the Shell Portion Due to Dead Load

[See Table 13.1 given at the end of this chapter.]

We will consider the shell *OAB* shown in Figure 13.5 (page 185) for our discussion. It is possible to calculate the *membrane forces* by mechanics from arch analysis. The shell is considered as a series of arches parallel to the force edge and another set perpendicular to the free edge. We may assume that the arches perpendicular to the free edge, not being restrained do not carry much loads.

13.4.2 Action of Live Load

As in the case of all shell structures, live load is usually taken on the plan area. It can be assumed to be carried to the support by pure axial thrust T_y. Its value is calculated for every point x, y as follows.

The horizontal component of the axial thrust due to this load on a hypar shell with equal sides ($a = b$) is given by the formula,

$$T_y^H = \frac{(wa^2)}{(2h_y)} \tag{13.3a}$$

Hence, the axial thrust is given by the formula,

$$T_y = \frac{(T_y^H)}{\cos\psi} \tag{13.3b}$$

where $\tan\psi = (2h_y) \times (y/b^2)$. (For all points along the line y/b, the value of $\tan\psi$ will also be a constant.)

The effect of the total value of T_y due to dead load and live load can be evaluated from the above. Generally, membrane forces are small and compressive. Even if these stresses are small and do not theoretically need any reinforcement, nominal steel as prescribed by IS for shells should be provided for these shells.

[**Note:** As discussed in cylindrical shells adding live load to dead load and finding stresses by one operation will also give safe values for design.]

13.4.3 Design of Groin Arch

Groins are designed as arches spanning opposite ends. As already mentioned in the case of conoids, we design and detail the arch as a two hinged or fixed arch, depending on the layout of the shell. For small spans, it may be taken as a three hinged arch because of its small stiffness at the crown. To analyze the arch, we have to estimate what cross-section or portion of the shell acts with the line of the groin. Any of the following two methods can be used.

Method 1: In this method, to make a conservative estimate of the dimensions of the arch that act, we assume that on each side of the groin a portion of the shell equal to eight times the thickness of the shell acts with the groin. Another more theoretical estimate of the portion is $1.52\sqrt{rt}$ in which r is the average radius of intersection and t is the thickness of the shell. It should be noted that the cross-section of the arch rib varies as it depends on the slope at which the two adjacent shells intersect, the V section thus formed being most acute angled near the corner. At the crown, the ribs are shallow.

The loading on the arch can be calculated from the internal forces along the intersection and resolve them into vertical and horizontal forces in the plane of the arch. The difficulty in this method lies in finding the angles at the intersection to compute the above forces parallel to the arch.

Method 2: This method is indicated by Figure 13.4. In this method, the entire section of the shell is treated as a free body. In such a free body, the moment produced by external loads can be calculated easily as shown in Figure 13.4.

For example, in Figure 13.4, the moment at point C of the arch is the algebraic sum of the moment at that point C of the following forces: (i) loads w, (ii) the reaction V and H at the hinge at the springing of the arch and (iii) the moments of the internal forces T_y and S.

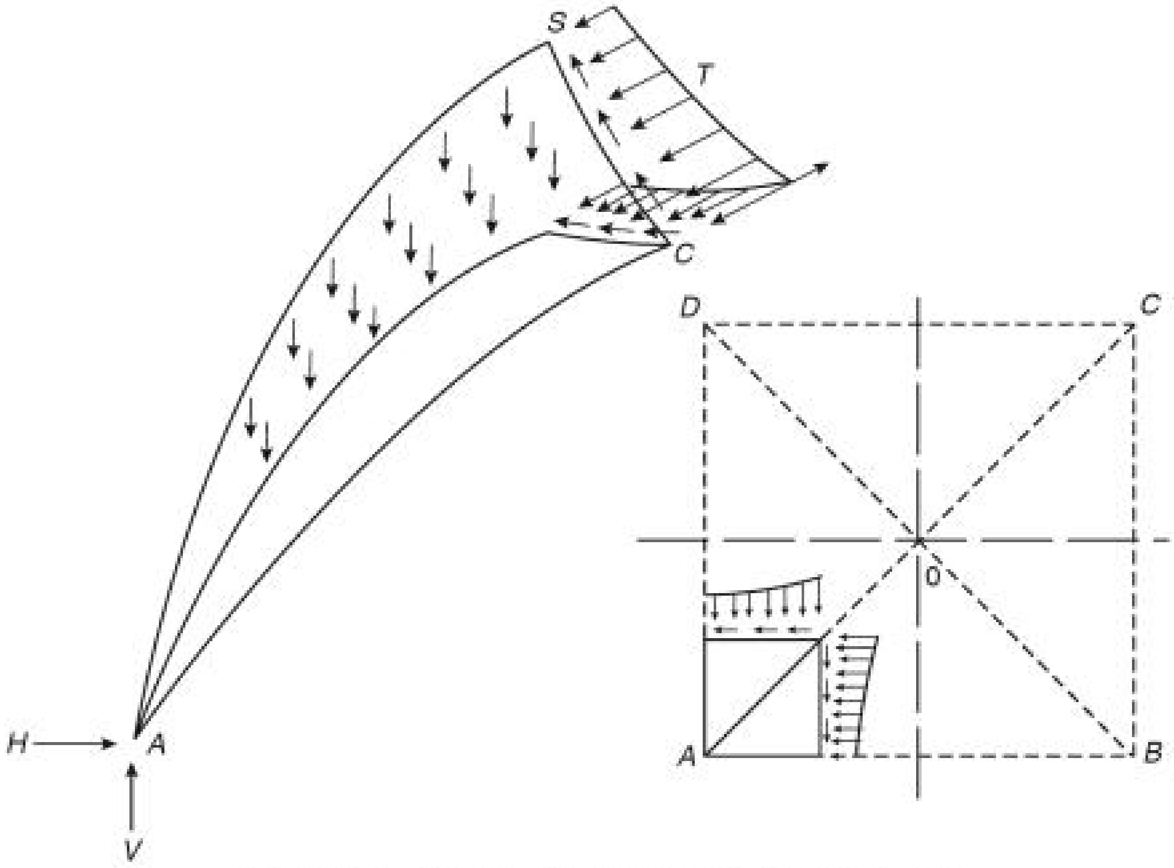

Figure 13.4 Design of groin arch of shells *AOB* and *AOD*.

The angle ψ, the T_y forces make with the horizontal, is obtained from the relationship [Eq. (13.1)],

We get

$$\frac{dz}{dy} = \tan\psi = \frac{(2h_y)y}{b^2} \quad \text{(for } T_y \text{ forces)} \tag{13.4a}$$

(As already stated, this is a constant for all points along a given y value.)

The angle ϕ the shear forces S make with the horizontal can be calculated from the formula [Eq. (13.1)],

We get

$$\frac{dz}{dx} = \tan\phi = \frac{(2h_x)x}{a^2} \quad \text{(for } S \text{ forces)} \tag{13.4b}$$

(This is constant for a given x value.)

With these forces, the arch can be analyzed. It is also very important that the detailing of steel should be properly carried out according to the standard practice for an arch.

13.5 ANALYSIS OF GROINED PARABOLIC (PARABOLOID) VAULTS

Parabolic vaults are similar to cylindrical shells formed by a parabola as directrix and straight line as generator (see Fig. 1.3). The coefficients from Table 13.1 are applicable only to *hyperbolic paraboloids formed by two parabolas of opposite curvatures.* If we make one of the curvatures zero, say $h_x = 0$, the component units are sections of *parabolic cylinders* or a paraboloid formed by one parabola.

When $h_x = 0$, the formula of Table 13.1 can be transformed for these parabolic shells as follows [2]:

$$T_x = \left(\frac{k_2 a^2}{16 h_y}\right) \times w_e k \left[\pi^2 \left(1 - \frac{x}{a}\right)^2 \cos\frac{\pi y}{2b}\right] \tag{13.5a}$$

$$T_y = -\left(\frac{k_1 b^2}{2 h_y}\right) \times \left(\frac{w_e}{k}\right)\left[\frac{k_2}{k_1}\cos\frac{\pi y}{2b} + 1\right] \tag{13.5b}$$

$$S = \left(\frac{k_2 ab}{4 h_y}\right) \times w_e \left[\pi\left(1 - \frac{x}{a}\right)\sin\frac{\pi y}{2b}\right] \tag{13.5c}$$

The definitions of the various symbols are the same as in Eq. (13.1). The same procedure of analysis as described for hyperbolic paraboloid shells can be used for these paraboloid shells also.

13.6 NATURE OF VARIATION OF MEMBRANE FORCES IN HYPERBOLIC PARABOLOID VAULTS

Generally, the T_x, T_y and S forces tend to be small. The direct forces T_x and T_y are also generally compressive. An inspection of Table 13.1 (refer end of chapter) gives us an idea of the variation of the membrane forces.

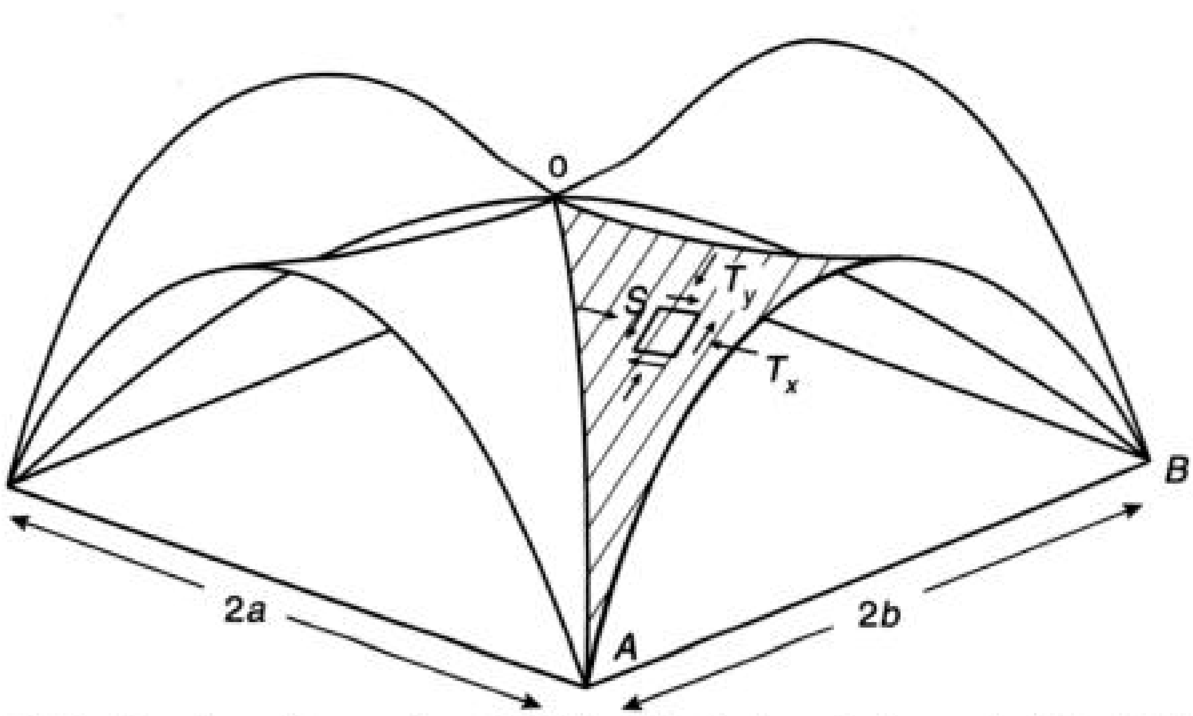

Figure 13.5 Membrane forces given in Table 13.1 for hyperbolic paraboloid shell *AOB*.

(a) *T_x forces.* T_x is maximum at the origin and decreases along X-axis. It becomes zero at the edges and along the edge, there can be no T_x forces. In Y-direction, T_x forces decrease with y/b values.

(b) *T_y forces.* T_y force is maximum at the edge $x/a = 1$ and $y/b = 0$. Along X-axis, it tends to decrease to the origin with decreasing x/a values. Along Y-axis, it tends to decrease with increasing value of y/b and becomes zero at $y/b = 1$.

(c) *S forces.* S forces are zero along $y/b = 0$ for all values of x/a. For other values of y/b, it tends to increase along X-axis. In each x/a section, the S value increases with y/b values. In practice, the maximum value will be obtained around $x/a = 0.4$ and $y/b = 0.4$ in most cases.

SUMMARY

This chapter gave a summary of the method of analysis of hyperbolic parabolic groined shells as proposed by Parme [2] by the use of published tables. The layout and concept of load transfer is very efficient. For small sized structures, a conceptual analysis of the layout using arch action would give satisfactory results.

EXAMPLE 13.1 [Design of a groined hyperbolic paraboloid shell] [2]
A groined hyperbolic paraboloid roof is 30 m × 30 m in span, with the maximum height of the main parabola $h_y = 11.25$ m. The rise of the centre of the main arch from the centre of the shell is, $h_x = 1.8$ m. The thickness of the shell is 100 mm. Assume dead load of the shell as 3.0 kN/m^2 and uniform live load as 1.5 kN/m^2. Calculate the internal forces in the shell. (Ratio $h_y/h_x = 6.25$) [Reference [2] may be consulted for more details of this problem.]

Reference	*Step*	*Calculations*
Eq. (13.1d)	1.	*Compute constants for the shell Fig:* The large parabola has $h_y = 11.25$ m and its axis along Y-axis = b The small parabola has $h_x = 1.5$ m and its axis along X-axis = a Any point on the shell is represented in Table 13.1 as y/b and x/a expressed as, $\bar{x} = \left[1 - \frac{x}{a}\right]\sqrt{\frac{h_x}{h_y}}$ First, we calculate $\bar{x}$ for $\frac{x}{a} = 0, 0.2, 0.4, 0.6, 0.8$ and 1 $\bar{x}$ for $\frac{x}{a} = (1-0) \times \left[\frac{1.8}{11.25}\right]^{\frac{1}{2}} = 0.40$ Similarly, $\bar{x}$ for, $\frac{x}{a} = 0.2;\ \bar{x} = 0.32$

Reference	*Step*	*Calculations*
		(b) $T_y = -\dfrac{k_1 b^2}{2h_y}\left(\dfrac{w_e}{k}\right)\left(1 + \dfrac{k_2}{k_1} \times \text{Coeff.}\right)$ $= \dfrac{-1.8 \times (15)^2}{2 \times 11.25}\left(\dfrac{3.0}{0.861}\right)(1 - 0.444 \times 0.7834)$ $= -40.3$ kN/m (c) $S = \dfrac{-0.8 \times (15)^2}{2(1.8 \times 11.25)^{1/2}} \times 3.0 \times 0.1462$ $= 8.67$ kN/m
Eq. (13.3a)	4.	*Calculation of stresses due to live load* We generally assume that live load (of 1.5 kN/m^2), which acts on the plan area, produces axial thrust only, i.e. by T_y forces (arch action). (Internal forces T_x and S are not affected by live load.) The horizontal component of T_y force designated as $T_y{}^H$ is given by the equation, $T_y^H = \dfrac{wa^2}{2h_y} = \dfrac{1.5 \times (15)^2}{2 \times 11.25} = 14.6$ kN/m (shallow shell) The actual axial thrust on the shell will be T_y,
Eq. (13.3b)		$T_y = \dfrac{T_y^H}{\cos\psi}$; value of $\cos\psi$ is found from ψ. For $\dfrac{y}{b} = 0.4; \tan\psi = \left(\dfrac{2h_y}{b^2}\right)y$ $= \left(\dfrac{2 \times 11.5}{15}\right)(0.4) = 0.6$; $\cos\psi = 0.857$ $T_y = -\dfrac{14.6}{0.857} = -17.04$ kN/m for y/b and $x/a = 0.6$ T_y for DL + LL = $-40.3 - 17.04 = -57.4$ kN/m [**Note:** If we had treated this live load of 1.5 kN/m^2 also as dead load, the value of T_x would have been –20.15 (half the value for dead load of 3.0 kN/m^2) compared to –17.04. Hence, for a quick design we may as well treat the live load also as dead load and proceed with the design.]
	5.	*Calculate maximum stresses* Calculation is repeated for other points and we find that the value of max. T_y occurs at $x/a = 1.0$ and $y/b = 1.0$ T_y due to DL = –93.57 T_y due to LL = –26.71 Total = –120.28 kN/m

Reference	*Step*	*Calculations*
		Stress = $\dfrac{-120.25\times 1000}{1000\times 100} = -0.12\ \text{N/mm}^2$
		This is low, similarly other stresses.
	6.	*Design the groin arch*
		The loads are transferred on the groin and we design the groin as a two pinned arch as described in Section 13.4.3.

TABLE 1 EXAMPLE 13.1 Coefficient *k* and Tabulation for Calculation of Internal Forces for Dead Loads (kN/m) **[2] [3]**

Forces	$x/a \rightarrow$	0	0.2	0.4	0.6	0.8	1.0
	$\bar{x} \rightarrow$	0.40	0.32	0.24	0.16	0.08	0
	$y/b \downarrow$						
	0.0	1.000	1.001	1.005	1.010	1.018	1.028
	0.2	– – –	0.959	1.962	0.968	0.975	0.985
k	0.4		– – –	0.861	0.866	0.873	0.882
	0.6			– – –	0.751	0.757	0.764
	0.8				– – –	0.652	0.658
	1.0					– – –	0.570
	0.0						
	0.2	– – –					
T_x	0.4		– – –				
	0.6			– – –			
	0.8				– – –		
	1.0					– – –	
	0.0						
	0.2	– – –					
T_y	0.4		– – –				
	0.6			– – –			
	0.8				– – –		
	1.0					– – –	
	0.0						
	0.2	– – –					
S	0.4		– – –				
	0.6			– – –			
	0.8				– – –		
	1.0					– – –	

[Reference 2, Table 2 gives more details of the procedure).

[**Notes:**

1. Points below the dotted lines are below the groin.
2. For $\bar{x}$, see step 1. When $x = 0$, $\bar{x}$ is maximum.

TABLE 13.1 *Contd.)*

$$S = \frac{k_2 ab}{2\sqrt{h_x h_y}} w_e (\text{Coeff.})$$

	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	0.10	.0000	.0098	.0196	.0293	.0389	.0483	.0576	.0666	.0754	.0838	.0920	.1016	.1106	.1190	.1266	.1334	.1394	.1445
	0.20	.0000	.0194	.0387	.0579	.0768	.0955	.1138	.1316	.1489	.1656	.1816	.2007	.2185	.2350	.2500	.2635	.2753	.2855
	0.30	.0000	.0285	.0569	.0851	.1129	.1403	.1671	.1933	.2187	.2433	.2669	.2948	.3210	.3452	.3673	.3871	.4045	.4194
	0.40	.0000	.0369	.0737	.1101	.1462	.1816	.2164	.2503	.2832	.3150	.3455	.3817	.4156	.4470	.755	.5012	.5237	.5430
S	0.50	.0000	.0444	.0886	.1325	.1759	.2185	.2603	.3011	.3407	.3789	.4156	.4592	.5000	.5377	.5721	.6029	.6300	.6533
	0.60	.0000	.0508	.1014	.1516	.2012	.2500	.2978	.3445	.3897	.4335	.4755	.5254	.5721	.6152	.6545	.6898	.7208	.7474
	0.70	.0000	.0559	.1117	.1670	.2216	.2753	.3280	.3794	.4292	.4774	.5237	.5787	.6300	.6775	.7208	.7597	.7939	.8232
	0.80	.0000	.0597	.1192	.1782	.2365	.2939	.3501	.4049	.4582	.5096	.5590	.6177	.6725	.7232	.7694	.8109	.8474	.8787
	0.90	.0000	.0620	.1238	.1851	.2456	.3052	.3636	.4205	.4758	.5292	.5806	.6415	.6984	.7510	.7991	.8421	.8800	.9125
	1.00	.0000	.0628	.1253	.1874	.2487	.3090	.3681	.4258	.4818	.5358	.5878	.6494	.7071	.7604	.8090	.8526	.8910	.9239

[See Section 13.6 for directions of membrane forces.]

REVIEW QUESTIONS

1. What are groined shells? Describe the mechanism of load transfer with which we can design these shells.
2. Describe the following types of shells:
   (a) Polygonal shells
   (b) Fanlight shells
   (c) Groined paraboloid shells
   (d) Groined hyperbolic paraboloid shells.
3. Clearly indicate by sketches the formation of hyperbolic paraboloid shells in groined vaults.
4. Indicate the two methods of design of the groin arch in groined vaults.

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14

DESIGN AND CONSTRUCTION OF A GROINED SHELL—AN EXAMPLE

14.1 INTRODUCTION

This chapter gives a brief account of the design of a groined shell roof, circular in plan and 22.5 m in (74 ft) diameter, that has been constructed for a church near Trichur in Kerala. The general layout is shown in Figure 14.1. Details of design and construction are given in Reference [1].

Figure 14.1 A general view of the Christ Church in Irinjalakuda, Kerala.

14.2 CHOICE OF SHELL ROOF

The conventional shell roof for a circular plan, namely, a spherical dome was ruled out for the church near Trichur from the point of view of architecture. The original proposal was to have the roof formed by anticlastic hyperbolic paraboloid groined shell units as shown in Figure 14.2, described in Chapter 3.

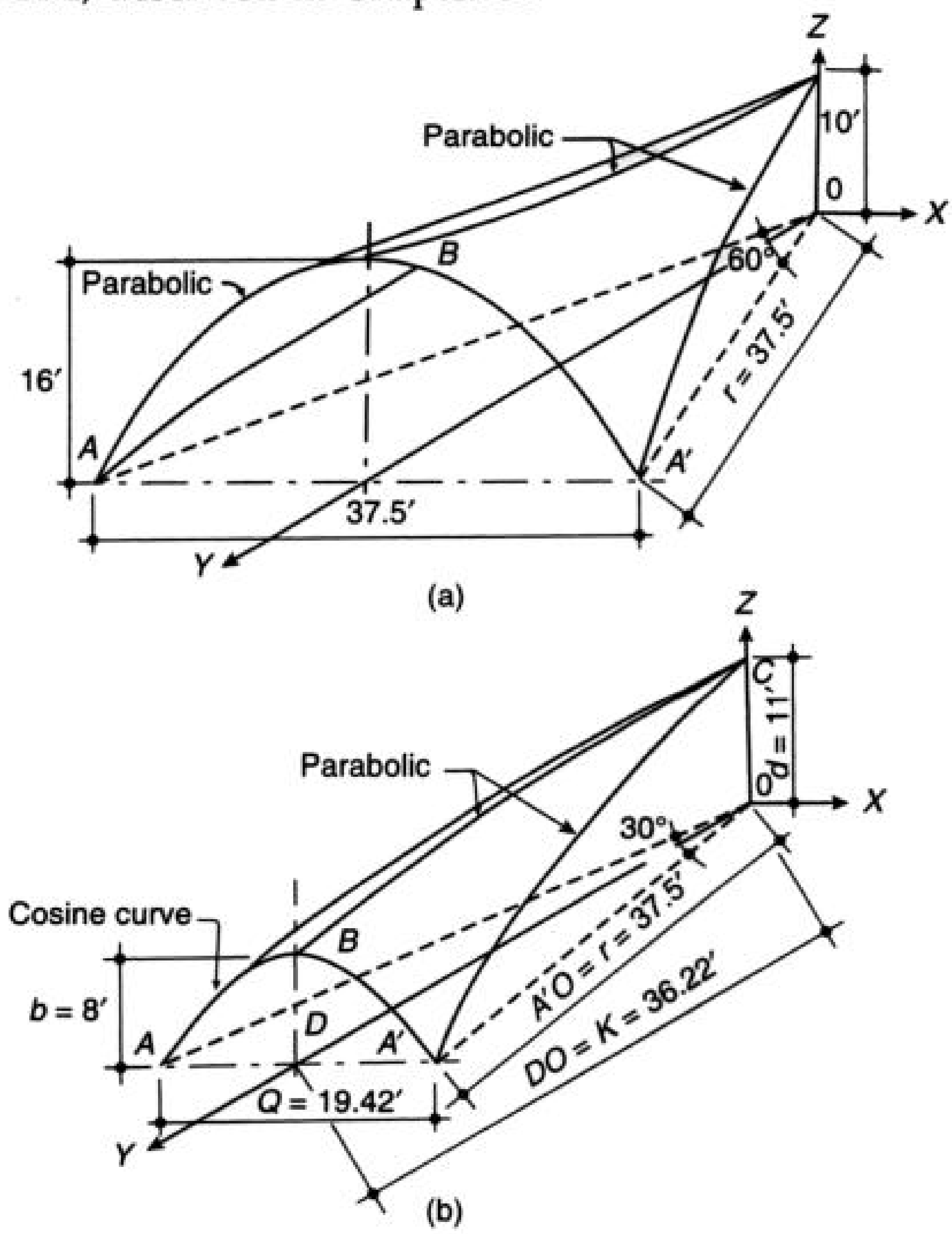

Figure 14.2 Alternatives considered for shell profile.

However, it was finally decided to adopt a groined shell of 12 divisions, each consisting of synclastic shells formed by a cosine wave form with its centre rising 8 ft (2.5 m) above the height at the periphery and a parabolic curve to the centre (a synclastic transitional shell) as shown in Figure 14.2(b). This provided a central rise of 3.35 m (11 ft) for the shell at the centre with a central rise to span ratio nearly equal to 1/7. This type of shell was chosen for the following reasons:

1. The number of 12 units was preferred by the church authorities.
2. A structure which "looked inwards" was preferred to one which "looked outwards", since in a church the emphasis was towards the inside.

3. The central cross would be located at the highest point of the roof, thereby making it conspicuous.
4. The shell is synclastic, which will perform better than anticlastic shells.

A cosine wave form was preferred to the circumferential curves because of aesthetics. (Reference [1] gives more details regarding design and construction of the groined shell.)

14.3 SHELL GEOMETRY

With the origin of coordinates at 0 as shown in Figure 14.2, the equation of the shell surface (one unit) can be written as,

$$z = d\left[1-\left(\frac{y}{k}\right)^2\right]+\left(\frac{y}{k}\right)^2 b\cos\left(\frac{\pi}{Q}k\frac{x}{y}\right) \tag{14.1}$$

where d = Rise at the centre of the shell = 11 ft
k = Length of the shell along Y-axis = 36.22 ft
b = Rise of cosine curve = 8 ft
Q = Span of cosine curve = 19.42 ft (column to column)

For the dimensions adopted for the shell, the equation of the shell surface reduces to the following equation in foot units:

$$z = 11.0-\left(\frac{1}{36.22}\right)^2\left[11-8\cos 1.866\,\pi\frac{x}{y}\right]y^2 \tag{14.2}$$

In actual construction, the bottom surface of the shell was made to conform to the above equation and variations in thickness at edges were made by adjusting the top profile. Each unit of the shell has a cantilevering portion at the periphery with a maximum overhang of 1.22 m (4 ft) in plan from the circle passing through the outer faces of the columns. The overhang is gradually reduced to 150 mm (6 in) near the columns.

14.4 BASIC STRUCTURAL ACTION

The structural action of the whole shell can be visualized for preliminary design purposes as follows. The circumferential curves (cosine curves) act as arches which transfer the load to the valley portion (groin). This assumption neglects the transfer of any shear (tangential and transverse) between each adjacent section. For uniform symmetrical loading, due to symmetry of structure and loading, these arches can be considered as fixed at the groins. All reaction components from two adjacent units, except the vertical reactions, cancel one another. The groin can be considered to be the main load carrying arch taking the load to the supporting columns. The supporting columns are relieved of the large horizontal thrust from these arches, by providing a *ring beam* capable of taking the horizontal reaction, and by effectively separating the ring beam from the columns with neoprene bearings as shown in Figure 14.3.

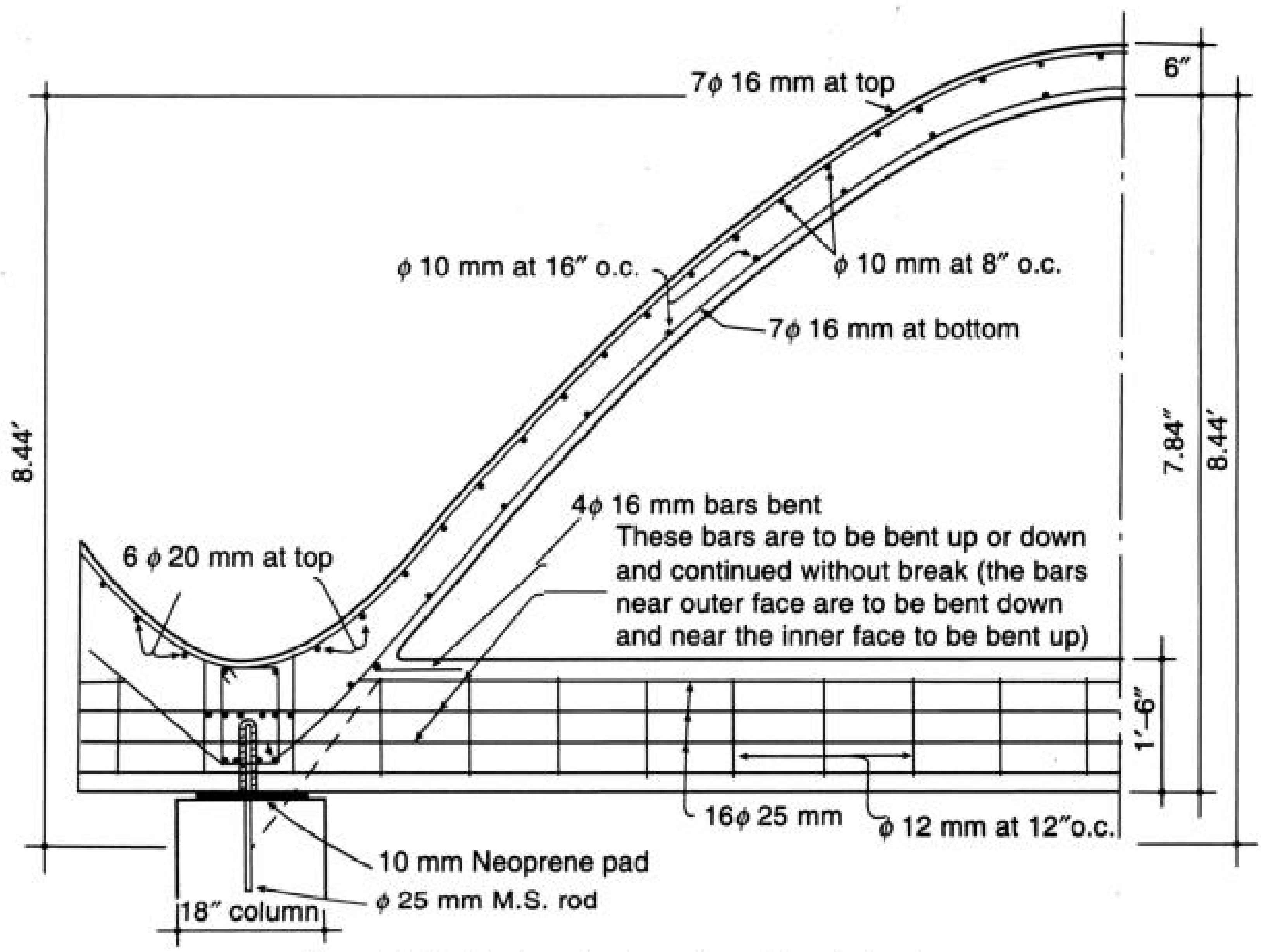

Figure 14.3 Section of end cosine arch and ring beam.

14.5 STRUCTURAL DESIGN

The shell roof was designed for self-weight and a superimposed load of 195 kg/m^2 (40 lb/ft^2) inclusive of finishes. A thickness of 75 mm (3 in.) was adopted since it was considered the minimum practical thickness for the site conditions. The reinforcement for the shell proper was governed by the minimum requirements stipulated by the codes rather than by the stresses. The general placement of steel is shown in Figure 14.4.

The effective width of the groin which acts as an arch was taken as 8 t on either side of the centre, t being the shell thickness. The thickness of the groin was increased from the centre of the shell towards the support as shown in Figure 14.5. The distribution of loading on the groin was idealized into a second degree parabola and the groin was analyzed for the following conditions:

1. Hinged at the two supports and at the centre; the moment of inertia of the arch at the centre is so small that it can be assumed to be hinged there.
2. Partially fixed at the two supports (fixing moment equal to the torsional resistance being offered by the ring beam), and hinged at the centre.
3. For support conditions 1 and 2 with one half of the arch subjected to dead load only, this condition is being included to take care of accidental unsymmetrical loads during decentering; the various sections of the groin were designed for the worst case of loading.

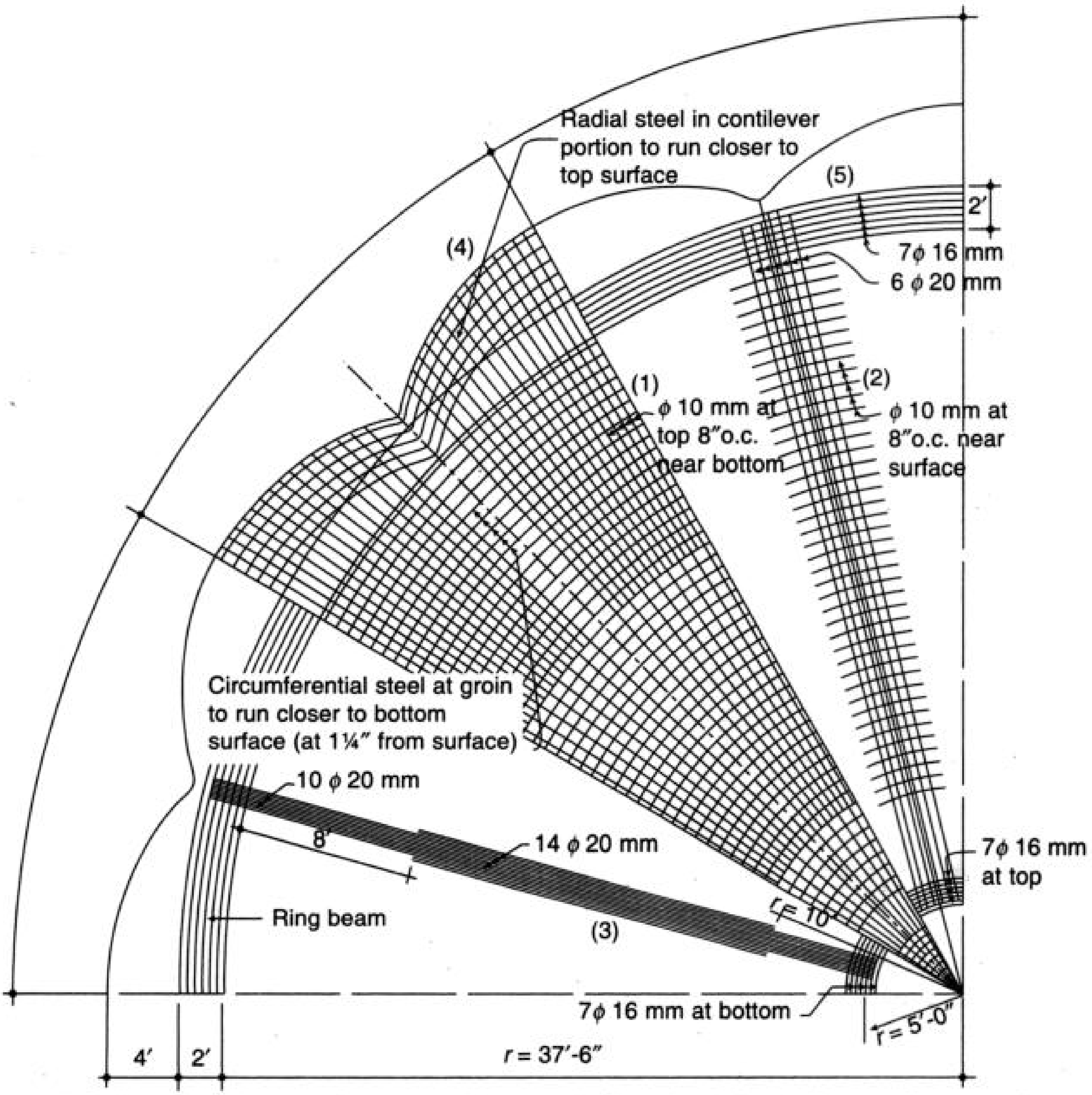

Figure 14.4 (1) Radial steel of 10 mm at 8"o.c. top and bottom, (2) Circumferential steel 10 mm at 8"o.c. near bottom, and (3) Groin arch steel 14 nos. of 20 mm.

The nominal reinforcement in the shell was sufficient to take the load on the overhanging portion of the shell designed as a cantilever.

Even though the thickness of the main shell is only 75 mm (3 in), the cosine arch at the circumference connecting the columns was thickened to 150 mm (6 in) as shown in Figure 14.3.

The ring beam shown in Figure 14.3 is subjected to the following forces:

1. Horizontal reaction from the shell which produces axial tension and bending moments of changing signs in the horizontal plane.
2. The fixing moment from the groin which induces torsion in the ring beam.
3. Self-weight of the ring beam which causes bending in the vertical plane and torsion.
4. The vertical reaction from the groin which is directly transferred to the column through the bearings.

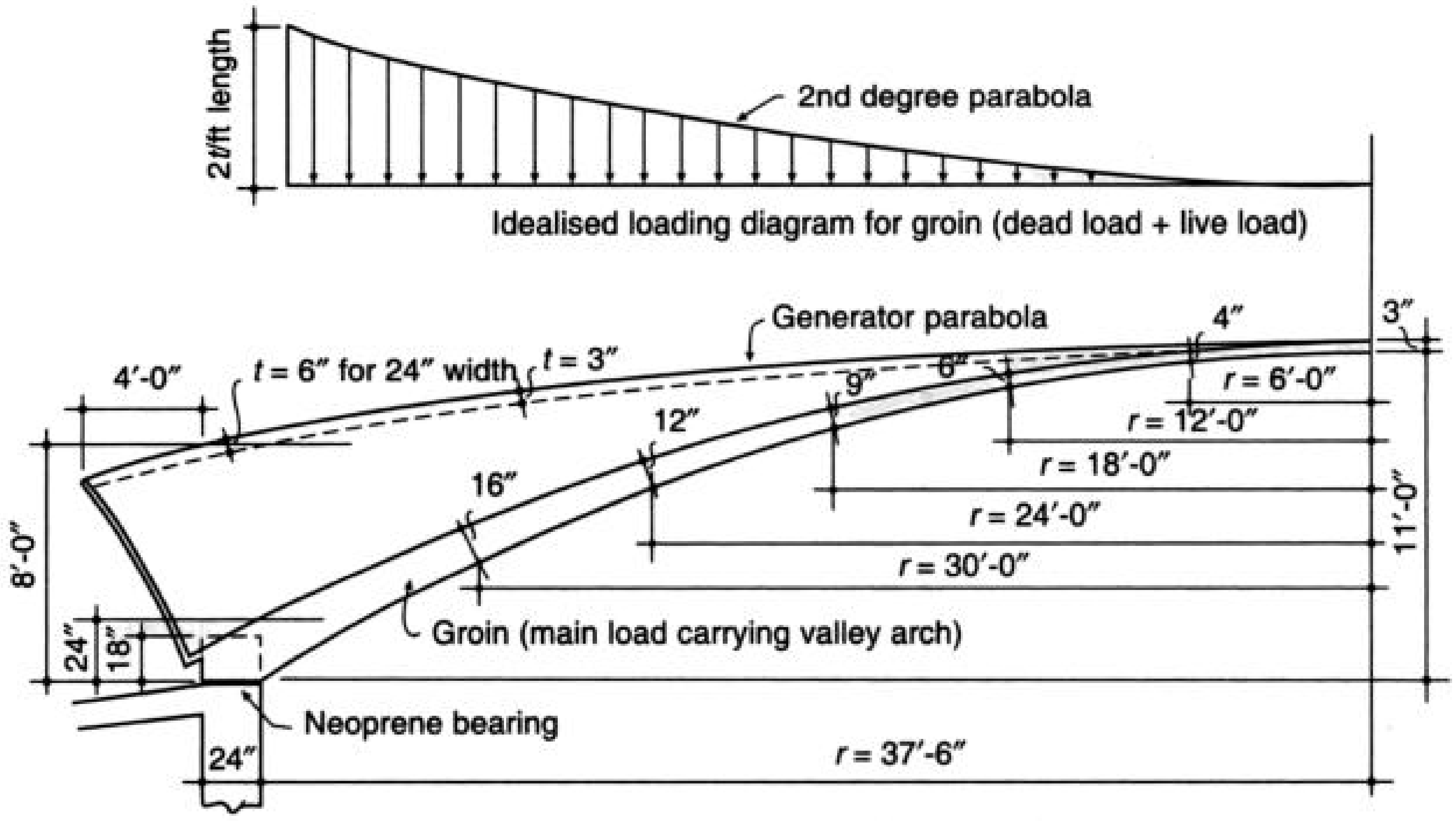

Figure 14.5 Longitudinal section and idealised loading diagram of groin.

The first of the above forces is the major one and the ring beam was designed for this force. Stirrups were provided to resist torsion and transverse shear. Critical sections were checked for combined stresses.

14.6 DETAILING OF SPECIAL REGIONS

It is important that edge members bordering the peripheries of the shell should be detailed with great care. Many failures have taken place because importance was given only to the calculation of stresses in the shell portion and less importance was given to detailing of these edge members. Hence, special attention was given for detailing the following regions:

1. *Groin to ring beam connection* (Figure 14.3): The most critical region in the shell is the junction between the groin and the ring beam. The whole load from the shell is transmitted to the ring beam through this junction. The following points were taken care of in detailing this junction:
   (i) The line of action of the thrust from the groin was made to pass through the point of intersection of the axis of the ring beam and the resultant vertical reaction from the column.
   (ii) All reinforcing bars from the groin were anchored well into the ring beam, and they were joined with those in the ring beam, by welding.
   (iii) The bars in the ring beam at this junction were made continuous by welding without allowing any break or lapping.
   (iv) Special care was taken in concreting this region to avoid any defect in concrete.
2. *Centre of the shell*: The general arrangement of steel in the shell is shown in Figure 14.4. If the reinforcement bars from all the groins (valley arches) are taken to the centre

of the shell, there will be overcrowding of steel in this region. To avoid this, the standard modifications in design and detailing were done as we usually carry out in the construction of domes.

(i) An effective ring beam was provided at the centre with a radius of 1.52 m (5 ft) from the centre, which could resist the horizontal reaction from the groins. This was achieved by providing extra steel in circles at this region, without any increase in the shell thickness [1].

(ii) A square grid pattern of reinforcement was provided at the centre and the bars were bent and continued in the radial direction for sufficient length to afford the necessary anchorage.

3. *Neoprene bearings at the column heads*: Neoprene bearings are characterized by their capacity to resist heavy vertical loads, while allowing sufficient lateral movement without developing appreciable lateral forces. They are used extensively for bridge bearings. Neoprene bearings were provided at the supports of the shell for the following two reasons:

(i) Even thought the ring beam is designed to resist the horizontal reactions from the groins (valley arches), the expansion of the ring beam due to this loading will cause outward movement of the columns if the column and the ring beam are cast together. Such movements will induce horizontal forces and *bending moments* in the columns. The neoprene bearings allow expansion of the ring beam without developing appreciable lateral forces on the column heads.

(ii) The expansion and contraction of the shell and the ring beam due to temperature changes are allowed, without causing similar movements in the columns.

10 mm neoprene pads with 1 mm steel plates bonded on to the two faces were used for the bearings. These are shown in Figures 14.3, 14.5 and 14.6.

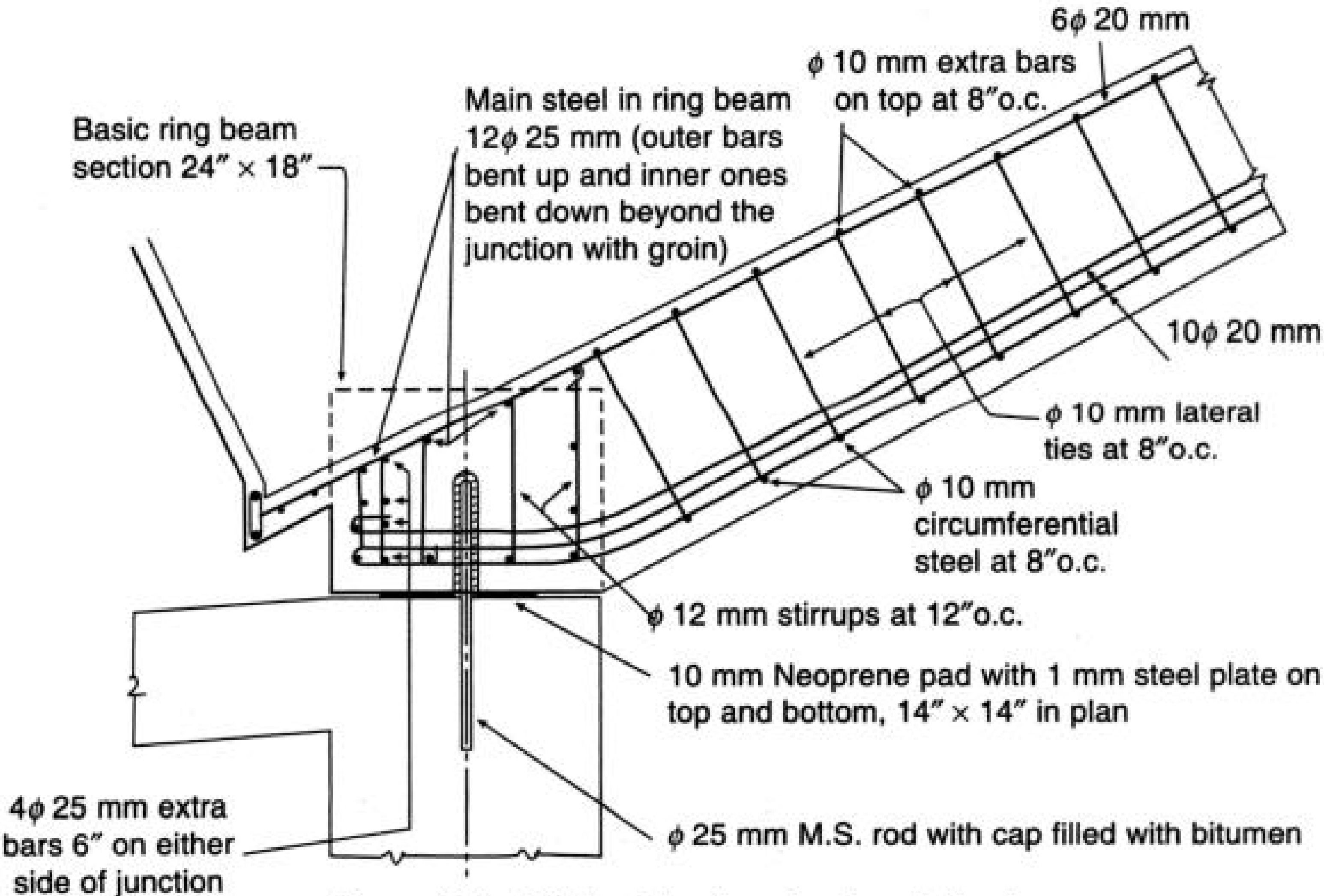

Figure 14.6 Details at junction of groin and ring beam.

14.7 CASTING THE SHELL

The formwork was built with bamboos and planks in three tiers (storeys). The reinforcements for the ring beam and the shell were assembled with great care. Conduits for electrical connections, nuts for hanging acoustic ceilings, bolts for fixing the cross, plugs for fixing conductor for lightning arrester and glazing at the open ends of the shell, and other fixtures were placed in position and secured firmly with the reinforcement grill. Since it was not possible to finish the concreting in one stretch, it was necessary to provide construction joints. Part of the groin (valley arch) was cast along with the ring beam to avoid a construction joint at the critical shell to beam junction. Standard code was followed to ensure proper location and correct forming of the construction joints.

It was necessary to provide back forms for some parts of the shell where the surface was very steep. These back forms consisted of 200 mm (8 in) wide planks tied on to the reinforcement, with sufficient cover. Only after concreting one 200 mm (8 in) width, the plank for the next 200 mm (8 in) width was introduced and held in position. The concrete used was grade M20. Concrete with aggregate of 10 mm or less maximum size was used in the region with crowded reinforcement near the centre of the shell.

14.8 CURING OF THE SHELL

Curing was done by spraying with water continuously for a period of 21 days. A hose from a nearby overhead tank was used for the purpose.

14.9 REMOVAL OF FORMS

The sequence of removal of formwork is very important in all shell construction. It should be gradual and in stages. All fixtures which held the reinforcements to the formwork were first removed. A sequence of gradual easing of the wedges *from the centre outward was followed*. After ensuring that the shell was off the formwork, the bamboo posts of the top tier were removed and the formwork dismantled and taken out. The bottom two tiers of staging were retained until all finishing works were done on the underside of the shell. The central deflection of the shell due to removal of formwork was less than 25 mm (1 in).

14.10 WATERPROOFING

A four-course waterproofing treatment with bitumen felt was given to the shell roof. It consisted of a first course of blown bitumen applied hot at 1.47 kg/m^2 (30 lb/100 ft^2) of area, a second course of roofing felt (hessian-base self-finished bitumen felt), a third course of hot bitumen and a fourth course of aluminum paint.

SUMMARY

This example has been included in this book to show how shells can be designed by "conceptual design" combined with theoretical analysis. As most of the shell surfaces are in compression or nominal tension, minimal steel compared to the large extent of the structure will be required. Recognizing the different types of forces that act in the different types of shells, detailing the pattern of reinforcement for these forces in these shells, proper detailing of edge members and providing proper load paths for the loads from the shell to be carried to the support are more important than the exactness of the design calculations for the shell. Special care should be given for the design and detailing of the important parts like junctions and the load carrying elements like the arch for the shell described in this chapter. References [1] to [4] are some of the references that helped the design and construction of this shell structure. (Late N.G. George was the architect for the project and the design and inspections of the words was undertaken by Indian Institute of Technology, Madras.)

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The simplest folded plate is the V-shaped unit; but it may not provide enough area of concrete at the top and bottom to resist the compressive forces due to bending and also to accommodate the needed reinforcement. Hence, the trough shaped or the trapezoidal units are more popular than the V-shaped units. Asymmetrical sections of Z-shapes can serve as northlight roof for factories [1].

The important reasons for the popularity of folded plates are: (i) They can span large distances like cylindrical long shells, (ii) They are cheaper than curved cylindrical shells because of their easiness in formwork (iii) They are more aesthetic in appearance, and (iv) The thickness of the slabs is small as in the case of shells (of the order of 1/200 of the span). Folded plates are studied along with shells under the heading "stress skin structures".

The three main parts of a folded plate are: (i) inclined plate, (ii) rigid joints between the plates, and (iii) transverse support or diaphragms at the ends. It is important to note that the joints are to be made rigid and should not undergo any change in their angles under the action of loads. It is due to this fact (that there should be no change in their joint angles) that we need correction analysis described in Chapter 16 to the preliminary analysis described in this chapter.

These folded plates cannot be classified as pure shell structures which take their strength from their curvatures. In folded plates, in addition to the transverse bending as a frame, longitudinal bending as a beam produces stresses similar to membrane stresses in shells. The reinforcements for these stresses are placed at mid-depth of the plates in the longitudinal direction.

In this chapter, we examine the historical development of folded plates and their preliminary analysis. In Chapter 16, we examine the corrections to be made in the preliminary analysis for angle changes between plates that will be present in the preliminary analysis to get the correct values of the stresses in the structure.

15.2 HISTORICAL REVIEW

It is interesting and instructive to study how the subject developed. The concept of folded plate construction started in Germany around 1925 with their use in deep coal bunkers as shown in Figure 1.5 (Chapter 1), where the bottom plates were built without any beams at the joints by just folding them. But it was only after World War II that American engineers applied the principles of folded plates for the construction of roofs. In coal bunkers, we deal with plates which are similar to deep beams. However, the American study was restricted to the study of plates which are not deep. In 1947, Winter and Pei published their paper on "stress distribution procedure" to folded plates in which the span of the plate is more than three times its breadth so that we can apply the ordinary beam theory of bending to these plates. Their approach was to make only *stresses* in the plates at the junction where the plates meet compatible. This we study in this chapter as *preliminary analysis* [2].

The above analysis proposed by Winter and Pei ignored the compatibility of deformation or rotation at the joints (change in angles at the junction of plates) and the corrections to be made, to keep the geometry of the joints. A number of methods for the

full analysis of folded plates, including the joint corrections mentioned above, are now available. In 1963, the American Society of Civil Engineers Task Committee published a review of these methods [3]. A full history of the study of the development of correction analysis can be made from this publication. Among the various methods of correction of the preliminary analysis. Simpson's method (1958) and Whitney's method (1959) are more popular than the others. We will deal with the correction analysis in Chapter 16 [3] [4].

15.2.1 General Dimensions

Folded plates can span large lengths with little materials and simple formwork. The thickness can be as little as 1/200th of the span. Their appearance is also good. Hence, they are becoming very popular for covering large spans.

As already stated in Section 15.1, there are many types of folded plates of which the most popular are the V-type for ordinary spans and the trough type for large spans. Northlight type and cylindrical shell type are used for special purposes. The V-type of folded plates is *not quite suited to large spans* as the thin slabs may not always provide the necessary concrete section to resist the compressive stresses and also for the placement of steel for bending in the transverse direction. Hence, we add horizontal slabs at top and bottom (usually about 40 cm in the middle and 20 cm at the ends) to make it into a trough type. Thus, trough type and cylindrical type are more practical type for use for large spans. As shown, Z-sections can be used as northlight roofs. It also has an inclined slab with horizontal slab at top and bottom. A series of simple units precast on the ground and then lifted up can also be used without use of elaborate formwork.

The folded plates are planned by taking into consideration the following points:

1. The inclination of the plates to the horizontal should not be more than 40 degrees for easiness in placing concrete. A slope below 30 degrees is considered too flat for action as a folded plate.
2. The width of the plate, in order to avoid deep beam action, should not exceed 1/3 the span. This width usually depends on the type of plates.

For a northlight Z, it may be as much as 1/3.3 span. For a V-type, we adopt a width of 1/5 to 1/6.5 of the span. For a trough type, it is usually only 1/10 to 1/12 of the span, the trough type being used for larger span than others.

The thickness of the plates is taken as not less than 1/200 the span, the minimum thickness being 100 mm. The horizontal parts of the trough and Z-types are usually made thicker than the inclined part. Thus, when the inclined part is 100 mm, we may take the horizontal part as 110 to 120 mm.

The best method to dimension folded plates is to adopt the dimensions of folded plates which have already been built successfully.

15.2.2 Methods of Analysis

Folded plates can be analyzed by the following methods:

- Beam method
- Elasticity method
- Slab beam analysis with correction analysis for stresses and rotation.

joint. The component from junction 2 (i.e. R_2) on plate 2 to junction 1 is called P_{21}. The component on plate 3 from junction 2 to junction 3 is called P_{23}. The resultant force on a plate is the sum of the forces from the junctions (one above and one below) of the plate. The load on plate 2 is P_2 (due to component of loads from joint 1 and joint 2).

Hence, $P_2 = P_{21} + P_{12}$ + Component of dead load along the plate.

(i) Stresses calculated on plates are *designated joint to joint*. Thus in plate 2, the stress at the top (junction 2 to junction 1) of plate 2 is compression f_{21} and the stress at the bottom of the plate 2 which is in tension is f_{12} [i.e. $f_{12} = -f_{21}$].

15.4.2 Description of Preliminary Analysis

(In the following descriptions, we must distinguish between the terms *procedures* and *steps* in each procedure.)

The following are the first three *procedures* in preliminary analysis described in Section 15.3:

1. Transverse slab analysis
2. Longitudinal beam analysis
3. Making compatibility of stresses

We will deal with the above three procedures for preliminary analysis in more detail in the following *eight steps*. (See in Example 15.3.) The following are the steps in preliminary analysis:

Step 1: Tabulate the dimensions of the folded plate.

Step 2: Tabulate the geometric properties of slab such as areas, etc.

Step 3: For transverse analysis assume the system as a frame with 1 m of the span as width and supported at each joint by vertical supports. Calculate loads and support moments for transverse analysis.

Step 4: Analyze the transverse frame by conventional moment distribution at supports (i.e. carry out transverse analysis of the frame). In moment distribution, the unbalanced moment is distributed in proportion to the stiffness I/L so that the moments on the two sides will be equal in magnitude but opposite in sign. The carry over factor to the other end of beam will be + 1/2.

We take a *unit width of the slab* in the transverse direction and support it by imaginary support at the joints for the analysis of this structure for UDL *by conventional moment* distribution. The spans theoretically should be the effective span between the supports, but not much error is introduced if the spans are taken centre to centre of supports. Complete the moment distribution and find the transverse bending moments and also the support reactions.

Step 5: Calculate the reactions R at the supports.

$$\text{Support reaction } = \frac{wd}{2} \pm \frac{\Delta M}{d\cos\phi} \tag{15.1}$$

where d is the width of the plate.

Step 6: Resolve joint reactions R into P loads acting along the plane of the slabs as shown in Fig. 15.4.

(Consider joint 2 – Let P_{21} denote load on plate from *joint 2* to *joint 1*.)

Next, we remove the support by applying equal and opposite reactions at the joints. This reaction should be shared by the plates meeting at the joint. Hence, resolve the reaction as forces acting in the planes of the plate as shown in Figure 15.4. Then for joint 2, we get (P_{21} means force from joint 2 to joint 1),

$$\frac{R_n}{\sin \alpha_n} = \frac{P_n}{\cos \phi_{n+1}} = \frac{P_{n+1}}{\cos \phi_n} \tag{15.2}$$

$$\frac{R_2}{\sin \alpha_{23}} = \frac{P_{21}}{\cos \phi_3} = \frac{P_{23}}{\cos \phi_2} \quad \text{(angles } \alpha \text{ and } \phi \text{ as in Sec. 15.4.1)}$$

(For plate 2.1) $$P_{21} = \frac{R_2 \cos \phi_3}{\sin \alpha_{23}} \quad \text{[i.e. cos } \phi \text{ of next plate]} \tag{15.3}$$

(For plate 2.3) $$P_{23} = \frac{R_2 \cos \phi_2}{\sin \alpha_{23}} \quad \text{[i.e. cos } \phi \text{ of previous plate]} \tag{15.4}$$

Resultant on plate 2 = $P_{21} - P_{12} = P_n$ (load on plate)

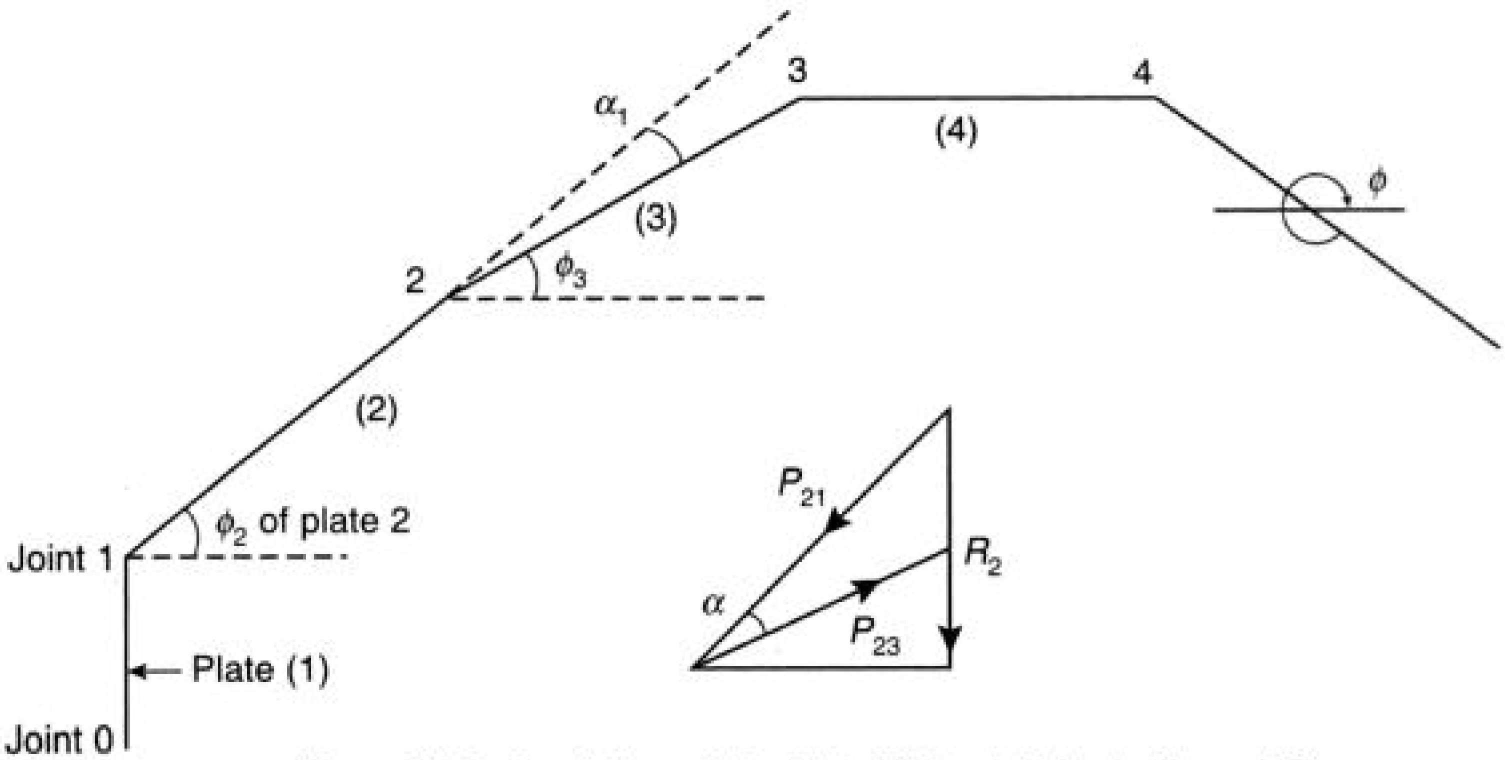

Figure 15.4 Resolution of R_2 at joint 2 to plate loads P_{21} and P_{23}.

Step 7: Longitudinal beam analysis—Determination of edge stresses.

The net inplane loads P_2 that we get by Step 6 on plates produce bending along the length of the folded plate by acting as a beam.

Bending moment on, say, plate 2 is given by,

$$M_2 = \frac{P_2 L^2}{8}$$

where P_2 is the resultant of inplane loads of plate 2 from joint above plate 2 (joint 2) and joint 1 below (joint 2).

The stresses at the extreme fibres of plate 2 are,

$$f_{12} = f_{21} = \frac{P_2L^2}{8}\left(\frac{1}{Z_2}\right)$$

where $Z_2 = \dfrac{td^2}{6}$.

Similarly, the stresses on plate 3 will be,

$$f_{23} = -f_{32} = \frac{P_3L^2}{8}\left(\frac{1}{Z_3}\right)$$

Step 8: Bringing stress compatibility by stress distribution procedure of Winter and Pei.

In theory, the stress on the same edge, say, f_{21} and f_{23} at joint 2 must be the same in magnitude and sign. But in the foregoing calculations (Step 2), they will not be equal. We correct this and equalize them as follows.

As explained in Section 15.3, Winter and Pei showed this can be carried out by *a method similar to conventional moment distribution*. In this method we assume that

1. The free edge stresses are considered as similar to fixed moments in moment distribution.
2. The difference in stresses is then distributed in proportion to $\left(\dfrac{1}{\text{Area of section}}\right)$,
   i.e. $\dfrac{1}{(\text{Depth} \times \text{Thickness})\text{ of plate}}$. The distribution should also result in the stresses at the two sides of the joint to be equal in magnitude and sign.
3. In the distribution, the carry over factor to the adjacent joint (the opposite end of plate) will be $-1/2$ (instead of $+1/2$ in Hardy Cross moment distribution).

The above procedures (1–3) carried out by steps 1 to 8 will give us the transverse moments as well as the final longitudinal stresses at junctions of the plate. [These results need further correction for compatibility of joint displacement, which we will examine in the next chapter.]

[This completes the approximate solution called preliminary analysis by Winter and Pei. But we know corrections are needed for compatibility of deformations, namely, no change in joint angles. The error depends on the configuration of the folded plate and the restraints of one plate over the deflection of the other. In general, the results of the corrections are not negligible and a complete analysis requires this correction which we will deal with in the next chapter.]

15.5 THEORY OF STRESS DISTRIBUTION METHOD FOR STRESS COMPATIBILITY

(This portion may be omitted on the first reading of this chapter.)

Consider two adjacent plates as shown in Figure 15.4. Take joint 1. The stresses obtained by preliminary transverse analysis on the adjacent plates plate 1 and plate 2 are different.

But they should be the same. This becomes possible due to the presence of additional shear at there joints which we will deal with later chapters. Hence, we introduce shear forces T_0, T_1, T_2 ... at junctions 0, 1, 2 ... to make them equal. Let $A = d \times t$ and $Z = I/y = td^2/6$.

Taking the plate 1, we have f_{01} as the stress in the top of plate 1. Similarly, taking plate 2, f_{12} is the stress at the bottom of plate 2.

$$f_{01} = \frac{T_0 - T_1}{A_1} - \left[(T_0 + T_1)\frac{d}{2}\frac{1}{Z_1}\right] + \frac{M_1}{Z_1}$$

$$\left(\text{As } \frac{d}{2}\frac{1}{Z_1} = \frac{d \times 6}{2 \times t \times d^2} = \frac{3}{A}\right)$$

$$f_{01} = \frac{-2T_0 - 4T_1}{A_1} + \frac{M_1}{Z_1}$$

Similarly, for stress at bottom of plate 2,

$$f_{12} = \frac{4T_1 + 2T_2}{A_2} - \frac{M_2}{Z_2}$$

As this should be equal in magnitude and sign, $f_{01} = f_{12}$, we get,

$$T_0\left(\frac{2}{A_1}\right) + T_1\left(\frac{4}{A_1} + \frac{4}{A_2}\right) + T_2\left(\frac{2}{A_2}\right) + \left[\frac{M_1}{Z_1} + \frac{M_2}{Z_2}\right] = 0$$

where A_1, A_2, Z_1 and Z_2 are constants and T forces vary as M. For UDL, M is parabolic.

This equation is similar to the well-known three moment equation, from which the Hardy Cross moment distribution is derived

$$M_A(l_1) + 2M_B(l_1 + l_2) + M_C(l_2) + \frac{6A_1\bar{x}_1}{L_1} + \frac{6A_2\bar{x}_2}{L_2} = 0$$

Hence, we adopt the following "stress distribution method" for equalization of stresses [6].

15.5.1 Winter and Pei Stress Distribution and Carryover Factors

In stress distribution, we have to make the stresses equal in *magnitude and sign*. We introduce the following important concepts [1]:

1. The effect of the shear force T_n which we introduce to equalize the stresses that meet at the joint is proportional to $(4/A_n)$ where A = depth of the plate. This is analogous to saying the stiffness factor for stress distribution is proportional to $1/A_n$ where A = (width × thickness) of plate. (Thus, the *distribution factor* will be inversely proportional to the areas and is to be carried out so that the stresses are equal in magnitude and sign.)
2. The effect of shear force T_n acting at junction n on the stresses of the plate at junction $n - 1$ is $(-2/A_n)$. Compared to $4/A_n$ (as in above), its value is $(-1/2)$, i.e. *the carry over factor is* $-1/2$. (Note the sign as different from Hardy Cross moment distribution where it is $+1/2$.)

CALCULATION

Step 1. Tabulate dimensions of plates (span 18 m)

TABLE 1, EXAMPLE 15.3 Dimensions of Folded Plates (Span = 18 m)

Plate No.	$\frac{\text{Plate width } (d)}{\text{Horiz. projection}}$ (m)	Inclination to horizontal (ϕ)	Angle between plates (α)	Thickness of plate h(m)
1	$\frac{0.2}{0.2}$	0	319–58′	0.1
2	$\frac{1.4}{1.072}$	40°–2′	40°–2′	0.1
3	$\frac{0.4}{0.4}$	0	40°–2′	0.1
4	$\frac{1.4}{1.072}$	319–58′	319–58′	0.1
5	$\frac{0.4}{0.4}$	0	319–58′	0.1

[**Note:** ϕ is the angle the plate makes with the horizontal. It is measured from plate line (extended to the next joint) to horizontal, clockwise. (See Figure 15.4.) α is the angle the plate makes with the next plate. It is measured by extending the plate and measuring the angle *to the next plate* measured clockwise.]

Step 2. Tabulate geometric properties

TABLE 2, EXAMPLE 15.3 Geometric Properties (in meter units)

Plate No.	Cross Sec. Area (m^2) (A_n)	Transverse analysis $I_T(10^{-4})$ (1 m *along span*)	Longitudinal analysis (Inclined width)	
			$I_L(10^{-4})$	$Z(10^{-3})$
1	0.02	0.833	0.67	0.67
2	0.14	0.833	228.67	32.67
3	0.04	0.833	5.33	2.67
4	0.14	0.833	228.67	32.67
5	0.04	0.833	5.3	2.67

Example:

$I_1 = 1 \times (0.1)^3/12 = 0.833 \times 10^{-4}$ (Thickness of plate 0.1 m)

$I_2 = 0.1 \times (0.2)^3/12 = 0.67 \times 10^{-4}$

$Z = I_1/(d/2) = 0.67 \times 10^{-4}/0.1 = 0.67 \times 10^{-3}$

C.O.			–3.6	+2.1	7.4	–12.7	–2.1	+2.1	+7.4	–7.4
Dist.			+3.6	–2.1	–7.4	+11.6	+3.2	–2.1	–7.4	+7.4
C.O.			–1.0	+1.8	5.8	–3.7	–1.0	+1.6	+3.7	–3.7
Dist.			+1.0	–1.6	–6.0	+3.7	+1.0	–1.1	–4.1	+3.7
Final moments	0	+6.8	–6.8	+44.8	–44.8	+32.7	–32.7	+34.5	–34.5	+34.5

[Notes:

1. In simple moment distribution, we find the difference between + and – moments at the joints and distribute the difference so that values at junctions are equal *in magnitude but opposite in sign.*
2. The final moments are in m.kg/m span.]

Example: In conventional Hardy Cross moment distribution, the moments at the joints from adjacent spans are to be of same magnitude but opposite sign. Hence in this method, we simply add the two with the sign and the result is distributed with opposite sign. Example, Joint 2, first line. Add 42.5 – 4.5 = +38. Distribute –38 in ratio 2/9 and 7/9 as –8.4 and –29.6 = 38. The resulting stresses are +42.5 – 8.4 = +34.1 and –4.5 – 29.6 = –34.1.

Step 5. Calculate reaction (R) at supports (See Table 5, Example 15.3)

$R = \dfrac{W}{2}$ (from left and right spans) (i.e. Items 1 and 2 in Table 5) i.e. load due to wt. +

$\pm \dfrac{M}{d_n \cos \phi_n}$ (from left and right spans) (i.e. Col, 3 and 4 of Table 5, Example 15.3) i.e. load due to moments

(It will be easy to arrive at the results by drawing a figure for each plate with the load and moments and calculate the values.)

TABLE 5, EXAMPLE 15.3 Calculation of reactions R at supports (kg/m)

Item	*Description*	R_0	R_1	R_2	R_3	R_4
1	From UDL—left span	0	68	238	68	238
2	From UDL—right span	0	238	68	238	68
3	$\dfrac{M(\text{left span})}{d_n \cos \phi_n}$	0	0	35.49	–30.28	1.707
4	$\dfrac{M(\text{right span})}{d_n \cos \phi_n}$	0	–35.49	30.28	–1.707	0
	Total reaction	0 = $R0$	270.5 = $R1$	371.9 = $R2$	274.0 = $R3$	307.7 = $R4$

Example. Item 1—R_2 from VDL plate 1 on left = 0.2 × 340 = 68 as cantilever.
Item 3—R_2 from moment on left span = (44.8 – 6.8)/1.072 = 35.49 kg.

Step 6. Calculation of P loads—Resolve R into P loads in plane of slabs for longitudinal bending

$$P_n = \left(\frac{-R_{n-1}}{\sin \alpha_{n-1}}\right)(\cos \phi_{n-1}) + \left(\frac{R_n}{\sin \alpha_n}\right)(\cos \phi_{n+1}) \qquad \text{[Eq. (15.2)]}$$

[Note: This expression for P_n is from R_{n-1} and R_n.]

Trigonometric values

Angle	sin	cos
40°–2′	0.6432	0.7664
319°–58′	–0.6432	0.7664

Example: (For Table 6, Example 15.3 below). Using the above formula,

1. Loads on plate 1—Using Eq. (15.2)

$$P_1 = 0 + \frac{R_1 \cos \phi_2}{\sin \alpha_1} = \frac{R_1 \times \cos 40°.2'}{\sin 319°.58'}$$

$$= \frac{270 \times 0.7664}{-0.6432} = -322.3$$

2. Load on plate 2

$$= \frac{-R_1 \cos \phi_1}{\sin \alpha_1} + \frac{R_2 \cos \phi_3}{\sin \alpha_2}$$

$$= \frac{-R_1 \cos 0}{\sin 319°.58'} + \frac{R_2 \cos 0}{\sin 40°.2'}$$

$$= \frac{-270.5}{(-0.6432)} + \frac{371.8}{0.6432} = 998.8 \text{ kg} \quad \text{(See Table 6, Example 15.3)}$$

TABLE 6, EXAMPLE 15.3 Calculation of *P* Values

Value of *P* is taken as positive when it acts right to left (kg/m span)

Plate No.	*From left support*	*From right support*	*Net load P*
Plate 1	0	322.3 (0 to 1)*	–322.3 (0 to 1)
Plate 2	+420 (2 to 1)	+578 (2 to 1)	+998 (2 to 1)
Plate 3	–442 (2 to 3)	+326 (3 to 2)	–116 (2 to 3)
Plate 4	–425 (3 to 4)	–478 (3 to 4)	–903 (3 to 4)

[Notes:

1.* Notation 0 to 1 indicates Joint 0 to 1.

2. Forces from higher joint to lower joint are +ve as they produce tension at the bottom of the plate. Forces from lower joint.

3. *P* forces are the results of the weight and live load from plates. Total *P* forces at end of plates = *P* × (*L*/2) on each plate (see Figure Ex. 15.1).]

Step 7. Longitudinal beam analysis and determination of edge stresses

[**Note:** We will work in kg.cm units in further calculations.]

$$M_{max} = \frac{P(18)^2}{8} = 40.5 \times P \text{ kg.m} = 4050 \times P \text{ kg.cm}$$

Find stresses in plate: $f_{12} = -f_{21} = M_{2(max)}/Z$

TABLE 7, EXAMPLE 15.3 Calculation of stresses at joints in kg/cm^2

Plate No.	M_{max} (kg cm) ($\times 10^3$)	*Z* (cm^3)	*f values at ends* (kg/cm^2)	*Joint*	*Stresses at joints on ... on opposite sides* (kg/cm^2)
1	–1305	–670	∓1956	1	+1956/+123.8
2	4045	32,670	±123.8	2	–123.8/–176.6
3	–471	2,670	∓176.6	3	+176.6/–112.0
4	–3659	32,670	∓112.0	4	+112/0
5	0	2,670	0		

[**Note:** *M* is in kg.cm units and *Z* is in cm units taken from Table 2, Example 15.3.]

Example: Plate No. 1 – $M_{max} = -1305 \times 10^3$ kg.cm; $f = 1305/670 = 1.95$

Step 8. Bring compatibility of stresses at joints by stress distribution by Winter and Pei method

The stresses at opposite sides of the folds are not equal. We introduce shear stresses to provide compatibility which has been shown can be carried out by a procedure similar to moment distribution with:

1. Distribution factors proportional to *inverse of areas* (depth × thickness of plates)

   $D_1 : D_2$ as $\frac{1}{A_1} : \frac{1}{A_2}$

   Present cases as thickness of plate *d* is same, it will be inverse of the width of plate. For joint1, $\frac{1}{0.2}$ to $\frac{1}{1.4}$, i.e. $\frac{7}{8}$ to $\frac{1}{8}$.

   For joint 2 = $\frac{1}{1.4}$ to $\frac{1}{0.4}$, i.e. $\frac{2}{9}$ to $\frac{7}{9}$ as shown.
2. The aim in this distribution is to make the stresses on either side *equal in magnitude and sign* unlike moment distribution where the moments are equal but opposite in sign.
3. Carry over factor = –1/2 (as against +1/2 in ordinary M.D.) [This is very important.]

   Example: For stress distribution, we proceed as follows (Refer Table 8 below). Taking joint 1, first line where both moments are +ve, the difference in stresses is 1956 – 123 = 1833. After distribution, we want them to be equal in magnitude and of same sign. We divide it as –1833 × 7/8 = –1603 to moment which is to be reduced and +1833 × 1/8 = +230 to moment that has to be increased as shown in Table 8, given below. We add these to the moments to be balanced so that the values are of same magnitude and sign, i.e. 1956 – 1603 = +353 and +123 + 230 = +353.

For joint 3 where one moment is +ve and the other negative, the difference is 176 –(–112) = 288. We have to distribute it so that the values are equal in magnitude and sign. Hence, we add $-288 \times 7/9 = -224$ to +176 making it equal to –48. We then add $+288 \times 2/9 = +64$ to –112 to make it –48, then making the stresses equal in magnitude and sign. For joint 2 where both moments are –ve, the difference is –53 and the signs are the same. Hence we distribute it as shown, add –12 and +41 making the both stresses equal to –135.

TABLE 8, EXAMPLE 15.3 Winter and PEI *Stress distribution* for compatibility of stresses at joints (kg/cm^2)

Joints	0		1		2		3		4	5
Plate	1		2		3		4		5	
Dist. F		7/8	1/8	2/9	7/9	7/9	2/9	2/9	7/9	
Stress Values	–1956	+1956	+123	–123	–176	+176	–112	+112	0	
Dist.		–1603	+230	–12	+41	–224	+64	–25	+87	
C.O.	+801		+6	–115	+112	–20	+13	–32	–44	
Dist.		+5	–1	+50	–177	+26	–7	–3	+9	
C.O.	–2.5		–25	+0.5	+13	–88	+1.5	+3.5	–4.5	
Dist.		+22	+3	–3	+10.5	–68	+19.5	–2	+6.0	
C.O.	+11		+1.5	–1.5	+34	–5	+1	–10	–3.0	
Dist.		1.3	–0.2	–8.0	–27.5	+4.0	–2	+1.5	–5.5	
Final values	–1145	+334	+334	–195	–195	–21	–21	+46	+46	

[Note: We call this operation as "stress distribution" and it is different from moment distribution.]

[Notes:

1. Free edges are considered similar to fixed edge of conventional moment distribution. Note beyond joint 4, we go to the symmetric centre line of the folded plate.
2. Unlike moment distribution, the stresses are to be the same in *magnitude and sign* on either side of the joint (unlike in regular bending moment distribution).
3. This solution proposed by Winter and Pei assumes that each plate is free to deflect and rotate without any restriction. But the adjacent plate restricts the freedom. Hence, the solution will not be correct unless we take into account the corrections described in the next chapter. The magnitude of correction will depend on the restriction of deformation between plates which will depend on the configuration of the folded plates. We will examine the correction calculations in Chapter 16. [This example is continued in Chapter 17 as correction analysis.]

EXAMPLE 15.4 [Preliminary analysis by Winter-Pei method]
Carry out preliminary analysis of a V-type folded plate for a span of 18 m shown in Figure E15.4. The thickness of horizontal slabs is 120 mm and that of inclined slabs, 100 mm. Assume applied live loading is 150 kg/m^2. The dead load varies with the thickness of concrete. (We will assume an easy slope of 31° and a width about 1/5 span, i.e. 18/5 = say 3.5 m giving a rise of the system as $L/10$ = 1.8 m.)

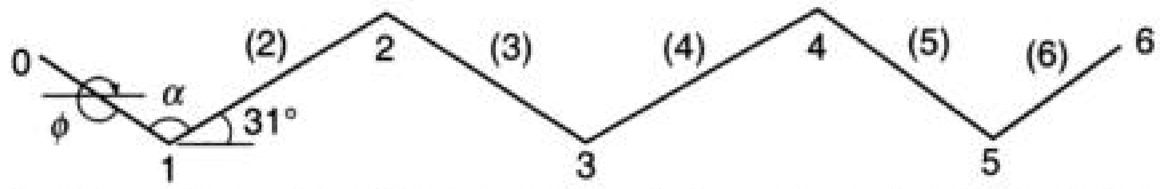

Figure E15.4 V type folded plate (joints denoted by 0, 1, 2, etc. and plates by (1), (2), (3), etc.).

PART I—PRELIMINARY ANALYSIS

TABLE 1, EXAMPLE 15.4 Dimensions of folded plates—Span—18 m (Step 1)
(Consider to centre line of the system)

Plate No.	*Width* (m)	*Plate width* / *Horiz. projection*	*Thickness* (m)	ϕ_n*(*degrees*)	α_n**(*degrees*)
1	2.10	$\frac{2.1}{1.80}$	0.12	329	298
2	3.50	$\frac{3.5}{3.00}$	0.10	31	62
3	3.50	$\frac{3.5}{3.00}$	0.10	329	298

Symmetric with respect to Junction 3

* $-\phi$ is the angle made by the plate to the horizontal. (See Step 1, Example 15.3.)
** $-\alpha$ is the angle between the plate and the next plate.

TABLE 2, EXAMPLE 15.4 Geometric properties of plates in metre units (Step 2)

Plate No.	*Cross Sect. Area* (m^2)	*Transverse analysis* $I_T \times 10^{-3}$ (m)	*Longitudinal analysis* $I_L(10^{-4})$	$Z_L(10^{-3})$
1	0.252	0.144	0.0926	0.0882
2	0.35	0.0833	0.3573	0.2042
3	0.35	0.0833	0.3573	0.2042

Example: I_L in plate 1 = $0.12 \times (2.1)^3/12 = 0.0926$

Z is in m^3. For conversion to cm units, multiply by 10^6 when we work in cm units (see Table 7).

TABLE 3, EXAMPLE 15.4 Loads and support moment for transverse analysis (Step 3)

(Consider unit metre along span)
(Insulation + LL) = 150 kg/m^2. DL for thickness of slab @ 2400 kg/m^3

Plate No.	*Total Load* kg/m *span*	*Horizontal span* (m)	*Support moment* (kg.m)
1	919.8	1.8	$\frac{918.8 \times 1.8}{2} = 827.8$
2	1365	3.0	$\frac{1365 \times 3.0}{12} = \pm 341.3$
3	1365	3.0	$\frac{1365 \times 3.0}{12} = \pm 341.3$

Example: Plate No. 1 – Load = [150 + (0.12 × 2400)] × 2.1 = 919.8 kg

TABLE 4, EXAMPLE 5.4 Transverse analysis by *moment distribution* as supported at joints (Step 4)

(kg and m units)

Joint 0 1 2 3

Plate	1	2		3	
Dist. factor	0	1	1/2	1/2	
Support moment	+827	–341 –486	+341	–341	+341
			–243 +122	122	
		61			61
Final values	827	–827	+202	–202	+410

TABLE 5, EXAMPLE 15.4 Calculation of reactions at supports (Step 5)

($R = W/2 + M/d_n \cos \phi_n$ from both sides)

Description	R_0	R_1	R_2	R_3
UDL from left span	0	919.8	682.5	682.5
UDL from right span	0	682.0	682.5	682.5
$M_{left}/d_n \cos \phi_n$	0	0	–208.4	69.4
$M_{right}/d_n \cos \phi_n$	0	208.4	–69.4	69.4
Total Reaction	0	1810.7	1087.2	503.8

$$R_n = \frac{W}{2} \pm \frac{M}{d_n \cos \phi_n} \qquad \text{[Eq. (15.1)]}$$

TABLE 8, EXAMPLE 15.4 *Stress distribution* for compatibility of stresses (Step 8)

Distribution in proportion to inverse of areas, i.e. $1/A$ and carry over $-1/2$

Joint	0	1	1	2	2	3
Plate	1		2		3	
Distribution		0.419	0.581	0.5	0.5	
Stresses	−80.7	+80.7	+55.8	−55.8	−49.9	+49.9
		−10.4	+14.5	−2.9	−3.0	
	+5.2		1.5	−7.3		1.5
		−0.6	+0.9	+3.7	−3.6	
Final values	−74.8	+68.8	+68.8	−57.0	−57.0	+53

Now joints which are not free (restrained) have to be corrected for rotation of joints by Simpson method given in Chapter 17. (The theory of the following calculations of part II is explained in the next chapter (16) and is based on Equations (16.1) and (16.2) of Section 16.3.2.)

Part II—Calculation of Plate Deflection of Plates from Data got from Preliminary Analysis (Span = 18 m)

(Theory given in next chapter.)(Refer Eg. 16.1)

$$y=\left(\frac{f_{(n-1)}-f_n}{9.6\times d_n}\right)\left(\frac{L^2}{E}\right)=\left[\frac{1800^2}{9.6\times 2\times 10^5}\right]\times\frac{f_{(n-1)}-f_n}{d_n}=\frac{1.688\left(f_{(n-1)}-f_n\right)}{d_n}$$

TABLE 9, EXAMPLE 15.4 Deflections in Preliminary Analysis

Plate No.	$f_{(n-1)}$	f_n	$f_{(n-1)}-f_n$	d_n	y = (cm)
1	−74.8	68.8	−143.6	210	−1.154
2	+68.8	−57.0	+125.8	350	0.607
3	−57.0	+53	−110.0	350	0.530

(This step is shown here to indicate that the deflections of the plate are usually extremely small. The procedure is detailed in the next chapter.)

REVIEW QUESTIONS

1. Give a short account of the development of the theory of modern analysis of folded plates.
2. What is meant by "transverse beam analysis", "longitudinal beam analysis" and "correction analysis" as used in the analysis of folded plate structures?

3. What are the length/width ratios prescribed for folded plates and what is the necessity for these limitations?
4. Explain the following: (a) "moment distributions used for transverse analysis" and (b) "method of stress distribution between joints" used in folded plate analysis.
5. Explain the Winter and Pei method of analysis of folded plates and state why the method needs further corrections to give a complete analysis.
6. In what way does the design of R.C. plates for coal bunkers differ from ACI design procedure of folded plates for roofs?

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16

FOLDED PLATES—CORRECTION ANALYSIS

16.1 INTRODUCTION

In the analysis of folded plates by the method proposed by Winter and Pei, explained in Chapter 15, we stopped with compatibility of stresses. But compatibility of deformation at the joints is also important and this has not been achieved in the preliminary analysis. The plate was allowed to bend freely without any constraint by the adjacent plate to which it is attached at the joints. Hence, this analysis in Chapter 15 can be taken only as a preliminary analysis and is incomplete. In this chapter, we examine the corrections to be made to the results of the preliminary analysis for the conditions that the compatibility of deformation of rotation of joints is also satisfied. We call this second analysis as *correction analysis*.

16.2 BRIEF SUMMARY OF CORRECTION ANALYSIS

The important condition to be satisfied in our analysis is that as the various joints of the plates of the folded plates are rigid, in actual field condition they do not undergo any change of angle when they are loaded. However, the preliminary analysis assumes that the plates are free to rotate which will cause *changes in the angles at the joints*. Hence, we must first find out what are the changes that have happened in the preliminary analysis and correct these changes. Correction analysis should ensure that there is no change in the angles at the joints. As already stated, several methods are available for this analysis. The three popular methods are:

1. *Whitney's method of analysis:* This method reduces the problem to a solution of simultaneous equations. This method is suitable for computation using modern computers [1].

Hence in a folded plate n,

$$M_{\max} = \left(\frac{f_{n-1} - f_n}{d_n}\right) \times (I).$$

Hence,

Deflection,
$$y = \left(\frac{f_{n-1} - f_n}{C_v d_n}\right)\left(\frac{L^2}{E}\right) = \pi^2 \text{ for} \tag{16.1}$$

$C_v = 9.6$ for a uniformly distributed load and $= \pi^2$ for a loading which varies as a sine function.

16.3.2 Equations for Plate Rotation from Deflections and Joint Rotation from Plate Rotation

As already stated in Section 16.3, we want the magnitude of joints rotation due to deflection of plates.

1. **Plate rotation:** Plate rotation is a result of beam deflections and hence it will be a function of deflection y.

   From the geometry of Figure 16.1, we can derive the equation for rotation of plates n and $n + 1$ as follows. (For details, see Reference [2].)

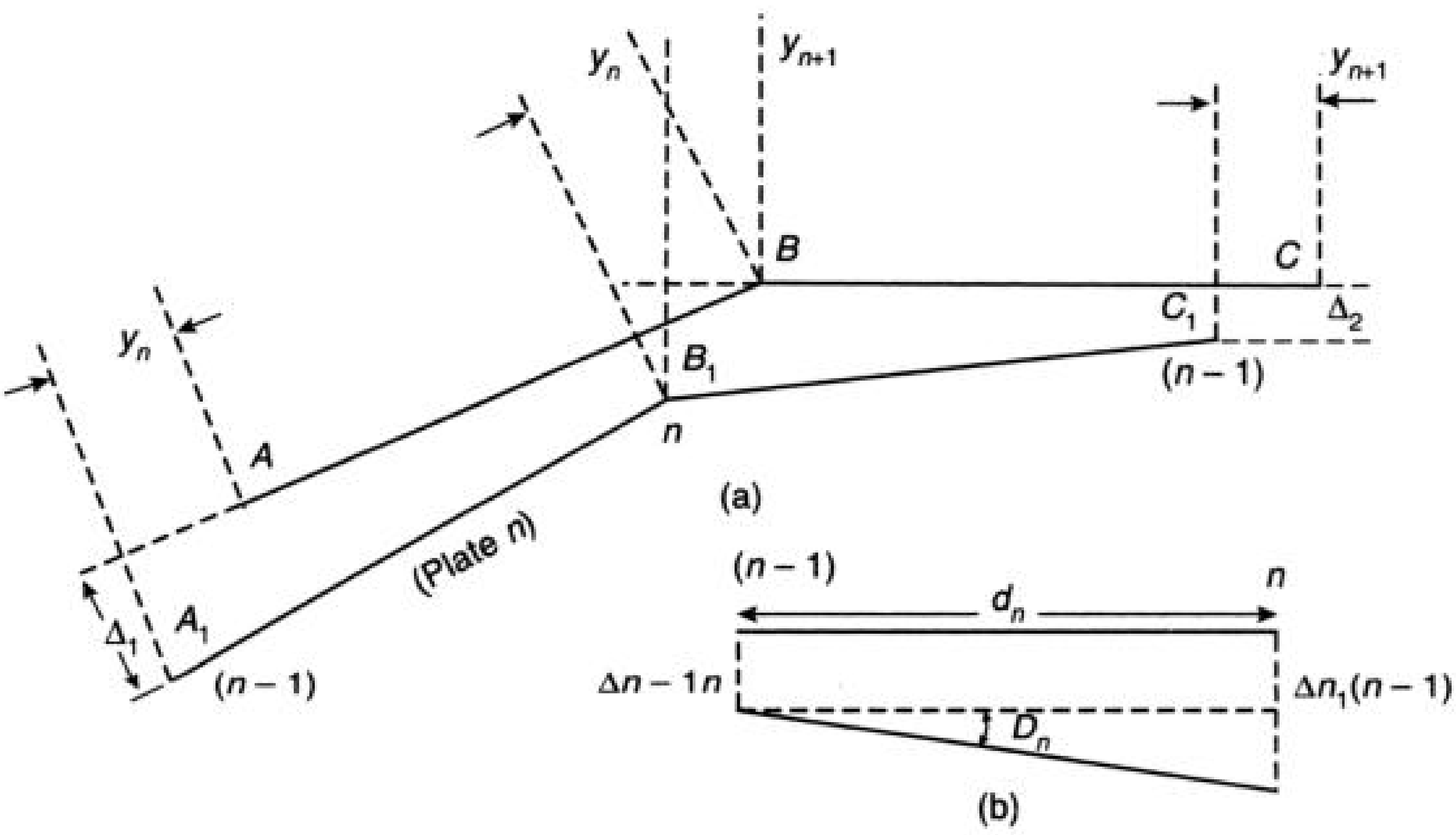

Figure 16.1 Plate rotations due to deflection of plates. (y. deflections along the plane of the plates and Δ deflections of edges perpendicular to plane of plates) Rotation of joint D_n.

The rotation of plate n designated as D_n will be a function of deflections of plates $n - 1$, n and $n + 1$ given by y_{n-1}, y_n and y_{n+1} and will be as follows [2]:

$$D_n = -\frac{1}{d_n}\left[\frac{y_{n-1}}{\sin \alpha_{n-1}} - y_n(\cot \alpha_{n-1} + \cot \alpha_n) + \frac{y_{n+1}}{\sin \alpha_n}\right] \tag{16.2}$$

Thus, the rotation of plate $n + 1$ will be,

$$D_{n+1} = -\frac{1}{d_{n+1}}\left[\frac{y_n}{\sin\alpha_n} - y_{n+1}\left(\cot\alpha_n + \cot\alpha_{n+1}\right) + \frac{y_{n+2}}{\sin\alpha_n}\right] \tag{16.2a}$$

[Clockwise rotation is taken as positive. The value of D_n is a function of three deflections y_{n-1}, y_n and y_{n+1}.]

2. **Joint rotation:** Joint rotation is given by the difference in *plate rotation* of plates joining at a joint. The notation we use for joint rotation is as follows. If we take joint n, the rotation due to preliminary analysis as 0, the rotation of that analysis is denoted by D_{n0}.

   (Later, we will be using unit moments at joints denoted by $X_1 = 1$ for joint 1, X_2 for joint 2, etc. The joint rotations caused by these moments for joint 'n' will be denoted as D_{n1}, D_{n2}, etc.)

   As the **joint rotation at joint** n due to **plate rotation** D_{n0} = Difference in plate rotations at the joint,

   $$\therefore \qquad D_{n0} = D_n - D_{n+1} \tag{16.3}$$

   For example, $D_{20} = D_2 - D_3$ [indicated in Eq. (16.2)]

   For correction analysis, we first find the rotations that will result due to the stresses in the plates we arrived at by Winter and Pei method. As there should be no rotation, we correct it by unit moment method described below.

16.4 EFFECTS OF APPLICATION OF UNIT MOMENT AT JOINTS

Correction analysis by unit moment method uses application of unit moment at joints taking one joint at a time. (We apply a moment of 1 m.kg/cm at a time which is equal to 100 cm.kg/cm). We will examine the effect of this moment on the plate rotations.

Let D_{20} be the rotation to be corrected from our preliminary analysis. We correct it by applying unit moments at the joints. Hence, we have to examine the effect of *application of unit moments at the joints*. It will have two effects.

As can be seen from Figure 16.2, it will produce in-plane forces which will cause joint rotation D, the same type we examined in Section 16.3. In addition, the unit moment will create rotation of joints which we call θ, which will be small and can be even neglected. Thus, the application of unit moment will have the following two effects as shown in Figure 16.2:

1. The **major effect** of application of unit moment will be joint rotation D produced by the inplane P forces. It will be as follows. As shown in Figure 16.2, the unit moment at joint n causes reaction (R forces) at that joint and also at the two joints adjacent to it. These R forces at these joints ($n - 1$, n and $n + 1$) will produce in-plane forces in the plates (P forces), stresses and rotation of the plates as we found in the preliminary analysis. Let us designate these again by letters D_n corresponding to *rotation* of plate n; D_n rotation of joint due to application of unit moment $X_2 = 1$ at joint 2. It can be calculated in the same way as we calculated D_{n0} in Section 16.3.

2. We will have another **minor** rotation due to beam action. It is the rotation of the joint considering bending in the transverse direction. We will denote this by symbol θ. The value of this rotation can be found by Eq. (16.4) given below. *Usually, this value is small and can even be neglected.* (The equation for θ is derived as shown below.)

[**Note:** The following two sections give the theory which may be omitted in the first reading of the text.]

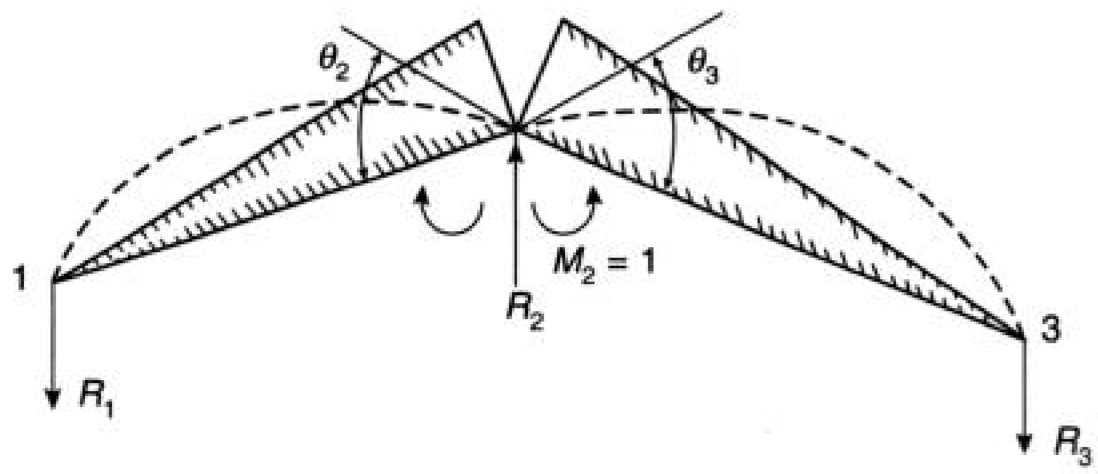

Figure 16.2 Two-fold effects of application of unit moment at joint (support reactions and joint rotation (–) which is usually small).

16.4.1 Evaluation of Plate Rotation Due to Unit Moment

When we apply unit moment at joint it produces reactions R at the supports as shown in Fig. 16.2. These reactions produce P forces in the plates. The rotation of these plates due to these forces can be determined as in the same way as we found it in our preliminary analysis. From plate rotations we can determine joint rotation also for each joint. This rotation is the major rotation for unit moment.

16.4.2 Equation for Joint Rotation θ Considering Beam Action Due to Unit Moment at Joint n

[**Note:** This joint rotation is an additional rotation due to unit moment. It is not a major factor and can be omitted in the first reading of the text and students can proceed to Section 16.6.]

Let us take effect of unit moment as shown in Figure 16.2, where we apply unit moment at joint n. The joints at $n - 1$, n and $n + 1$ undergo rotation θ. Its value can be obtained by the second theorem of "moment area theorems" (formulated by Otto Mohr in 1868) which states that the tangent deviation is equal to moment of the bending moment diagram.

Let d_n and d_{n+1} be the width of the plates and unit moment be applied at joint n. We denote the rotation of joint $n - 1$ due to unit moment at n as $\theta_{n-1,n}$. Assuming h_n as the thickness of the plate,

$$\theta_{n-1,n} = \frac{\text{Moment of BM diagram}}{(EI)\times(d_n)}$$

$$= \left[\frac{1}{2}d_n \times \frac{1}{3}d_n\right] \div \left(\frac{Eh_n^3}{12}\right) \times d_n = \frac{2d_n}{Eh_n^3}$$

Hence, $$\theta_{n-1,n} = \frac{2d_n}{E(h_n)^3} \tag{16.4a}$$

$$\theta_{n,n} = \frac{4d_n}{E(h_n)^3} + \frac{4d_{n+1}}{E(h_{n+1})^3} \tag{16.4b}$$

and $$\theta_{n+1,n} = \frac{2d_{n+1}}{E(h_{n+1})^3} \tag{16.4c}$$

In general, the value of θ will be very small and usually need not be considered in our calculations except for very exact analysis.

16.5 TOTAL JOINT ROTATION DUE TO UNIT MOMENT

The total joint rotation is due to D effect and θ effect, i.e. $D'_{nx} + \theta_{nx}$. [For all practical purposes, we may neglect θ_{nx}.]

We will first take the D effect or the rotation of the plates due to the P forces produced by application of unit moment as stated in Section 16.3.2, item (2). We call it D'_{nx}.

According to Eq. (16.3), the rotation of plate produces the rotation of joint. Denoting the rotation of joint n due to application of moment $X = 1$ and cause P forces as D'_{nx}, we get,

$$D'_{nx} = D_n - D_{n+1} \tag{16.5}$$

For example, when $X_2 = 1$, we have

$$D_{n2} = D_n - D_{n+1} \text{ (of plates due to } X_2 = 1)$$

Total rotation due to application of unit moment: For the total rotation of the joints, we have to add θ values also to the above joint rotation values.

$$\text{Total rotation, } D_{nx} = D'_{nx} + \theta_{nx}$$

Thus, for joint rotation due to application of $X_3 = 1$ at joint 3, we will have for joint 2,

$$D_{23} = D'_{23} + \theta_{23} \tag{16.6}$$

[As already stated, as θ is usually small, *we may delete it in our calculations*.]

16.6 DESCRIPTION OF CORRECTION ANALYSIS

Based on the theory explained above, we adopt the following procedure. As already stated, correction analysis is a continuation of the "preliminary analysis" described in Chapter 15. The preliminary analysis accomplishes only stress compatibility. Correction analysis brings in joint rotation compatibility also. The following procedure is commonly used for correction analysis.

Part I—We carry out the preliminary analysis as described in Chapter 15 by steps 1 to 8. These eight steps in the Winter and Pei method are (see Section 15.3):

Step 1: Tabulate details of plate.

Step 2: Tabulate geometrical properties.

Step 3: Tabulate loads and moments for transverse analysis.

Step 4: Carry out moment distribution.

Step 5: Tabulate R forces.

Step 6: Tabulate P forces.

Step 7: Tabulate stresses (f_1 and f_2) at ends of plates.

Step 8: Carry out stress compatibility.

Part II—Calculation of joint rotation from results of preliminary analysis (D_0 values)
Continue preliminary analysis to find the resulting rotation of joints which we have to cancel by correction analysis. This is carried out in the following three steps:

Step 9: Calculate and tabulate the deflection caused by the stresses obtained in primary analysis using Eq. (16.1).

Step 10: Calculate plate rotations, D_n using Eq. (16.2). (We designate the rotation to be cancelled at joint n as D_{n0} as calculated in the next step.)

Step 11: Calculate the joint rotations using Eq. (16.3). These are the joint rotations we have to cancel D_{n0}. (Note the rotation.)

(This is the end of the primary analysis.)

Part III—Unit moment method for correction analysis
Apply unit moment $X_2 = 1$, $X_3 = 1$, etc. at each joint successively *one by one*. (Usually, the first plate is free to rotate and hence the first joint we take for unit moment application is joint 2, $X_2 = 1$.)

[**Note:** For convenience in calculation instead of $X = 1$ m kg/m we work with $X = 100$ cm kg/cm. The resulting values are multiplied by 100 to get m kg/m]

Item 12: (a) Calculate transverse beam rotation θ using Eq. (16.4). [As already stated, this will be a small quantity and may be omitted for a rough calculation.]

(b) Calculate R forces due to unit moment and repeat steps 5 to 11 of preliminary analysis (called items 5 to 11 indicated here) of Chapter 15 and find joint rotation, D values.

Item 13: Add θ and D to find total rotations of joints due to application of unit moments at a joint. (Just as D_{n0} denotes rotation to be corrected obtained from preliminary analysis, D_{nm} denotes rotation at n due to X_m equal to unity applied at joint m.)

Part IV—Solution of simultaneous equations
D_{n0} is the rotation we get after Winter and Pei preliminary analysis.

There should be no rotation of the joints as the joints are rigid. Hence, we write down the condition that the rotation of each joint including the rotation in primary analysis should reduce to zero. (Stated otherwise, the rotations produced by X_2, X_3, etc. should cancel the rotation of the preliminary analysis.)

For joint 2, $X_2D_{22} + X_3D_{23} + X_4D_{24} \ldots D_{20} = 0$ (16.7)

For joint 3, $X_2D_{32} + X_3D_{33} + X_4D_{34} \ldots D_{30} = 0$

From these, we can solve X_1, X_2, etc. and also correct the Winter and Pei stresses and get the real stresses for calculating the reinforcements required as follows.

Part V—Find "corrected final design values" of transverse moments and f_1 and f_2 values for midspan

In step 4, we carried out the transverse analysis in the Winter and Pei method. This transverse moments are used for the design of transverse steel and thickness of plate. To the results of the transverse moments obtained by preliminary analysis, we have to add the effects of the value of the joint moments X_1, X_2, etc. obtained by correction analysis to get exact values. These give the design moments at the joints.

Similarly, the stresses f_1 and f_2 are used for the design of longitudinal steel in the slabs. The final values of f_1 and f_2 for design are obtained by taking the results of the preliminary analysis and adding to them the effect of the values of X_1, X_2, etc. obtained in the correction analysis.

These values are for the midspan. For other places, say, for quarter span, they can be obtained as follows.

Part VI—Calculation of design moments and stresses for sections along the length of the folded plate other than midspan [4]

As already stated both the preliminary analysis and correction analysis are made for midspan. For the application of these results for the other sections we may assume following as shown in Tables 16.1 and 16.2 given below.

1. The *transverse moment of the preliminary analysis* is constant along the span.
2. The *longitudinal stresses of the preliminary analysis* vary parabolically.
3. The *effects of the correction analysis*. The moments and stresses produced by X values can be assumed to have sine variation along the span.

TABLE 16.1 Design moments at any point along the span (Say at quarter plan)

Joint	*Preliminary Analysis*	*Effect of Correction Moment*
1	Constant along span	
2	Sine variation along span	
3		

TABLE 16.2 Design longitudinal stresses along span (say at quarter span)

Joint	*Preliminary Analysis*	*Effect of Correction Moment*
1	Parabolic variation along span	
2	Sine variation along span	
3		

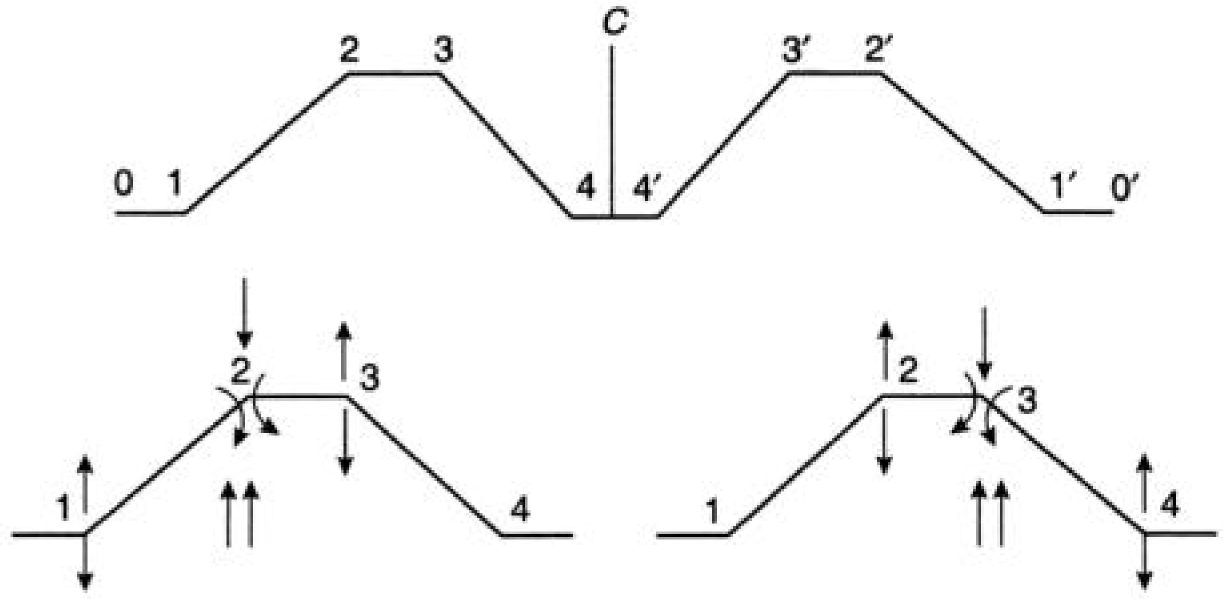

Figure E16.1 Application of unit moments at joints 2 and 3 simultaneously.

Continuation of Preliminary

Step 9. Calculate deflections from stresses calculated in Chapter 15 [Eq. (16.1)]

$$y_n = \frac{(f_{n-1} - f_n)}{(9.6)(d_n)}\left(\frac{L^2}{E}\right) = \frac{1.688(f_{n-1} - f_n)}{d_n}$$

L = 1800 cm, $E = 2 \times 10^5$ kg/cm^2 and f in kg/cm^2.

TABLE 9* Calculation of plate deflections y from preliminary analysis (We use kg and cm units for easiness of calculation)

Plate No.	(kg/cm^2)		d_n (cm)	y_n (cm)
	f_{n-1}	f_n		
1	–1145	+334	20	y_1 = –124.30
2	+334	–195	140	y_2 = +6.38
3	–195	–21	40	y_3 = –7.37
4	–21	+46	140	y_4 = –0.80
5	+46	+46	40	y_5 = 0

*This problem is a continuation of Example 15.3 of Chapter 15 which ended in Table 8. Hence we mark continuation of Table 8 of Example 15.3 as Table 9; d_n = width of plate n; y = deflection.

Step 10. Calculate plate rotations (D_n) [Eq. (16.2)]

$$D_n = -\frac{1}{d_n}\left[\frac{y_{n-1}}{\sin\alpha_{n-1}} - y_n(\cot\alpha_{n-1} + \cot\alpha_n) + \frac{y_{n+1}}{\sin\alpha_n}\right]$$

Working in centimetres,

$$D_2 = -\frac{1}{140}\left[\frac{-124.3}{\sin 319.58} - 6.38(\cot 319.58 + \cot 40.2) + \frac{(-7.37)}{\sin 40.2}\right] = -1.299$$

$$D_3 = -\frac{1}{40}\left[\frac{6.38}{\sin 40.2} - (-7.37)(\cot 40.2 + \cot 40.2) + \frac{0.80}{\sin 40.2}\right] = -0.656$$

$$D_4 = -\frac{1}{140}\left[\frac{-7.37}{\sin 40.2} - (-0.8032)(\cot 40.2 + \cot 319.58 + 0)\right] = 0.0818$$

$D_5 = 0$ (see Table 10)

Step 11. Calculate joint rotations (D_{n0})

For joint 'n', it is $D_{(n)} - D_{(n+1)}$

$$D_{20} = D_2 - D_3 = -1.299 - (-0.656) = -0.643$$
$$D_{30} = D_3 - D_4 = -(0.656 - 0.082) = -0.738$$
$$D_{40} = D_4 - D_5 = 0.0818 - 0 = +0.0818$$

TABLE 10 Plate Rotations and Joint Rotations in Preliminary Analysis

Plate	*Plate Rotation*	*Joint*	*Joint Rotation*
2	$D_2 = -1.299$	2	$D_{20} = -0.6434$
3	$D_3 = -0.656$	3	$D_{30} = -0.7374$
4	$D_4 = +0.0818$	4	$D_{40} = +0.0818$

F the joint rotation is positive, it means enlargement of the angle at the joint.

Part IIIA Correction Analysis Joint 2

Correction analysis for $X_2 = 1$: Apply unit moment at joint 2 (X_2 also 100 cm.kg/cm **or** 1 m.kg/cm **for easiness of calculation)**

[Notes:

1. As plate 1 is a cantilever, joint 1 is free to rotate. Hence, first apply unit moment to joint 2. $X_2 = 1$ m.kg/m or 100 cm.kg/cm.
2. *We apply correction moments* 1 m.kg/cm *span for convenience of calculation. Also, this we put as* 100 cm.kg/cm *span. This should be specially noted.*]

[**Note:** As already stated, **the steps** corresponding to the preliminary of correction analysis are designated as **items**.]

Item 12: **Calculate "beam" rotations (a) θ [Eq. (16.4)] and (b) D for X_2** = 100 cm.kg and rotation D.

[We will work with kg and cm units.]

$$d_2 = 140 \text{ cm}; \; d_3 = 130 \text{ cm}; \; h = 10 \text{ cm}$$

(a) Beam rotation θ

Since joint 1 is free, no rotation is developed. With joint 2, as quantities are small, apply $X_2 = 100$ cm.kg/cm span length [From Eq. (16.4)]

$$\theta_{22} = \frac{4 \times 100 d_2}{E h_2^3} + \frac{4 \times 100 d_3}{E h_3^3} = \frac{400}{2 \times 10^5 \times (10)^3}(140 + 40) = 0.00036 = 0.036 \times 10^{-2}$$

$$\theta_{32} = \frac{200 d_3}{E h_3^3} = \frac{200 \times 10}{2 \times 10^5 (10)^3} = 0.004 \times 10^{-2}$$

(These values are small compared to D values got from deflection as given below due to X_2 = 100 cm.kg/cm.span.)

TABLE θ values for X_2 = 100 cm.kg/cm

Item	*Joint 1*	*Joint 2*	*Joint 3*
X_2 = 100 cm.kg/cm	0	0.036×10^{-2}	0.004×10^{-2}

[**Note:** *These values are small compared to the joint rotation values got from deflection as given below and hence need not be taken into account for rough analysis.*]

[**Note:** As the next steps correspond to steps 5 to 11 of preliminary analysis, the next corresponding "steps" in this analysis will be referred as "items". Thus, item 5 corresponds to step 5 of preliminary analysis.]

Calculate beam rotation D **for** X_2 **= 100 cm.kg due to** P **forces.**

Item 5: **Reactions** R **at joints are on application of moment** X_2 = 100 cm.kg/cm [Eq. (15.1)]

$$R_1 = \frac{-100}{d_n \cos \phi_n} = \frac{-100}{140 \cos 40.2} = -0.9322 \text{ kg/cm run}$$

$$R_3 = \frac{-100}{40 \cos \phi_3} = \frac{-100}{40} = -2.5 \text{ kg/cm run}$$

$$R_2 = 2.5 + 0.9322 = 3.4322 \text{ kg/cm run}$$

[**Note:** If we were to use X_2 = 1, the resultant values would have been small.]

Item 6 **(same as step 6 in preliminary analysis)—Resolve** R **into** P **forces**

Resolve R_1, R_2, R_3, etc. due to in-plane loads P_1, P_2, P_3, etc. and repeat stress determination stress compatibility as in Chapter 10. Taking ϕ as angle,

$$P_n = -R_{n-1}\left(\frac{\cos \phi_{n-1}}{\sin \alpha_{n-1}}\right) + R_n\left(\frac{\cos \phi_{n+1}}{\sin \alpha_n}\right)$$

R_1 will affect pates 1 and 2—$P_1 \times P_{2-1}$.

R_2 will affect plates 2 and 3—$P_{2-2} \times P_{3-1}$.

R_3 will affect plates 3 and 4—$P_{3-2} \times P_4$.

(These will be kg/cm run along span.)

Find stresses due to P_1, $(P_{2-1} + P_{2-2})$, $(P_{3-1} + P_{3-2})$ and P_4.

$$P_n = \left(\frac{-R_{n-1}}{\sin \alpha_{n-1}}\right)\cos \phi_{n-1} + \left(\frac{-R_n}{\sin \alpha_n}\right)\cos \phi_n$$

TABLE FOR ITEMS 5 AND 6 Values of R and P for X_2 = 100 cm.kg/cm run

R values (kg/cm run)	*R values* (kg/cm run)
$R_1 = -0.9322$	$P_1 = 1.110$
$R_2 = 3.4322$	$P_2 = 3.88$
$R_3 = -2.5$	$P_3 = -7.07$
	$P_4 = 3.89$

(The P values are given in Table above.)

Item 7 **(same as step 7 in preliminary analysis)—From the table above, determine bending moment and find stresses f_{10} and f_{20}**

$$f = \pm P_n\left(\frac{L^2}{8}\right)\left(\frac{6}{h(d_n)^2}\right) \quad \text{for UDL, where } L,\ h \text{ and } d \text{ are in cm } [h = \text{thickness}]$$

[The value of $Z = hd^2/6$.]

We adopt a sinusoidal loading, hence instead of $\left(\frac{L^2}{8}\right)$ we use, $\left(\frac{L^2}{\pi^2}\right)$.

$$f = \pm\frac{(1800)^2}{\pi^2}\left(\frac{6}{10\times(d_n)^2}\right)\times P_n = \pm 196800\frac{P_n}{(d_n)^2}$$

TABLE ITEMS 6 AND 7* *P* forces and *f* values for X_2 = 100 cm.kg/cm along span

Plate	*Value of* P(kg/cm.run)	f_1(kg/cm^2)	f_2(kg/cm^2)
1	1.11	+546.1	–546
2	3.88	+39.0	–39.0
3	–7.07	–869.4	+869.4
4	3.89	+39.0	–39.0

*Items 6, 7 are similar to steps 6 and 7 of Chapter 15.

Item 8 **(same as step 8 in preliminary analysis)—Make stress compatibility (for X_2 = 100 cm.kg)** (Distribution factor 1/Area) (Carry over factor –1/2)

TABLE ITEM 8* Stress compatibility for X_2 = 100 kg.cm/cm (Similar to step 8, Chapter 15)

Joint 0		1		2		3		4	
D.F.		7/8	1/8	2/9	7/9	7/9	2/9	2/9	2/9
Stress	+546	–546	+39	–39	–869	+869	+39	–39	
Dist.		+512	–73	+184	+646	–646	+184	+8.7	–30.3
C.O. –1/2	–256		+92.3	+36.5	+323	–323	–4.4	–92	15.2
	+259.8	+28.6	+28.6	–139.3	–139.3	+153	+153	–68	–68

D.F. = Distribution factor; Dist. = distribution; C.O. = Carry over.

Item 9 **(same as step 9 described in Section 16.6 in preliminary analysis, part II)—Calculate deflection of plate and tabulate [Eq. (16.1)]**

TABLE ITEM 9 Calculation of deflections X_2 = 100 kg.cm/cm

Plate No.	d_n (cm)	f_{n-1}	f_n	$(f_{n-1} - f_n)$	y (cm)
1	20	+259.8	+28.6	+231.2	+18.95
2	140	+28.6	−139.3	+167.9	+1.97
3	40	−139.3	+153.0	−292.3	−11.98
4	140	+153.0	−68.0	+221.0	+2.59
5	40	−68.0	−68.0	0	0

$y = 1.688\ (f_{n-1} - f_n)/d_n$ (See Part II—Step 9 at start of Example 16.1.)

Items 10 and 11 (same as steps 10 and 11 in preliminary analysis, part II, Section 16.6)—Plate rotations

Plate	*Plate Rotation*
2	$D_2 = 0.3438$
3	$D_3 = -0.8904$
4	$D_4 = 0.1330$

Joint	*Joint Rotation*
2	$D'_{22} = D_2 - D_3 = 1.2342$
3	$D'_{32} = D_3 - D_4 = -1.0234$
4	$D'_{42} = D_4 - D_5 = 0.1330$

(Item 12 was carried out at the start of part III.)

Item 12 from page 235 and 13 **(same as step 12 in preliminary analysis) – Calculate total joint rotation** $D + \theta$

Total joint rotation of joint 2 = Sum of the (rotation of the beam + rotation of slab) (Steps 4 and 5)

$$D_{22} = 0.036 \times 10^{-2} + 1.2342 = 1.23456$$
$$D_{32} = 0.004 \times 10^{-2} + (-1.0234) = -1.02336$$
$$D_{42} = 0.1330$$

TABLE ITEM 13 Total rotation of joints for X_2 = 1

Joint rotation	*Transverse beam rotation* θ *from Step* (1)	*Rotation from deflection* (D) *from* Table 10	*Total rotation* ($\theta + D$)
D_{22}	0.036×10^{-2}	1.2342	1.23456
D_{22}	0.004×10^{-2}	−1.0232	−1.02336
D_{42}	—	0.1330	0.1330

[**Note:** We may neglect θ as it is small.]

Part IIIB Correction Analysis Joint 3

(Apply $X_3 = 1$**) Apply unit moment at junction 3** (X_3 = 100 cm.kg/cm **and repeat steps)**

Item 12. Find beam rotation θ_{23}, θ_{33} and θ_{34}

$$\theta_{23} = \frac{2 \times 100 \times d_3}{Eh_3^3} = \frac{200 \times 40}{2 \times 10^5 \times 10^3} = 0.004 \times 10^{-2}$$

$$\theta_{34} = \frac{2 \times 100 \times d_4}{Eh_4^3} = \frac{200 \times 140}{2 \times 10^5 \times 10^3} = 0.014 \times 10^{-2}$$

$\theta_{33} = 2(0.004 + 0.014) \times 10^{-2} = 0.036 \times 10^{-2}$ (This is twice sum of θ_{23} and θ_{34})
(These are also small and can be neglected.)

TABLE FOR X_3 = 1 (θ values for X_3)

	θ_{23}	θ_{33}	θ_{43}
$X_3 = 1$	0.004×10^{-2}	0.036×10^{-2}	0.014×10^{-2}

Item 5: **Find adjacent reactions R_2, R_3 and R_4**

$R_2 = -2.5$ kg/cm run
$R_3 = 3.43$ kg/cm run
$R_4 = -0.9322$ kg/cm run

Item 6: **Resolve into P forces and find stresses and carry out stress compatibility** ($X_3 = 1$)
(Plates affected on either side of joint 3, i.e. plates 2, 3 and 4, 5) Net values,

$P_2 = -3.887$ kg/cm run
$P_3 = 7.058$ kg/cm run
$P_4 = -3.887$ kg/cm run
P_5 = Due to symmetrical load, there will not be any load on P_5.

TABLE FOR ITEMS 5 AND 6 Values of R and P for $X_3 = 1$

R values (kg/cm)	*P values* (kg/cm)
$R_2 = -2.50$	$P_1 = -3.887$
$R_3 = 3.43$	$P_2 = 7.058$
$R_4 = -0.93$	$P_3 = -3.887$
	P_4 = Nil

Item 7: Find f_0 stresses due to P for $X_3 = 1$ and use distribution method for stress compatibility

TABLE FOR ITEMS 6 AND 7 P forces and f values for X_3

Plate No.	*P values*	f_1	f_2
1	–3.887	39	39
2	7.058	+868	–868
3	–3.887	–39	+39
4	Nil	–	–

Item 8: **Make compatibility of stresses** (X_3 = 100 cm.kg/cm run)

TABLE FOR ITEM 8 Stress compatibility for X_3 = 100 kg.cm/cm

0	1		2		3		4	
	7/8	1/8	2/9	7/9	7/9	2/9	2/9	7/9
		–39	+39	+868	–868	–39	+39	
	–34	+5	+184	–645	+645	–184	–8.8	+30
+17		–92	–2.5	–322	+322	+4.4	+92	–15
+46.5	–91.9	–91.9	+165.1	+165.1	–156.7	–156.7	+68	+68

Item 9: **Find deflection of plates as in Step 11.1 (y values) for** X_3 = 100 kg.cm/cm

TABLE FOR ITEM 9 Deflection calculations for X_3 = 100 kg.cm/cm

Plate No.	$f(n-1)$	$f(n)$	y (cm)
1	+46	–91	y_1 = 11.35
2	–91	+165	y_2 = –3.01
3	+165	–156	y_3 = 13.19
4	–156	+68	y_4 = –2.63
5	+68	+68	y_5 = 0

Items 10 and 11: **Find rotation of plates (D values) and joint rotations**

TABLE FOR ITEMS 10 AND 11* Plate and Joint Rotation D for X_3 = 100 kg.cm/cm

Plate No.	*Plate Rotation*
2	$D_2 = -0.0204$
3	$D_3 = +1.0016$
4	$D_4 = -0.1465$

Joint No.	*Joint Rotation (Slab)*
2	$D_{23} = D_2 - D_3 = -1.025$
3	$D_{34} = D_3 - D_4 = 1.1511$
4	$D_{45} = D_4 - D_5 = -0.1465$

*Similar to Table 10 for Chapter 16.

Item 12 was carried out at the beginning of Part IIIA.

Item 13: **Total rotation of joints (Beam + Plate) rotations** ($\theta + D$) **for** X_3 = 1 kg.cm/cm
Due to X_3 = 100 kg.cm (Note subscript 3)

$$D_{23} = 0.00004 - 1.025 = -1.025$$

$$D_{33} = 0.00036 + 1.1511 = 1.1514$$

$$D_{43} = 0.00014 - 0.1465 = -0.1463$$

TABLE FOR ITEM 13 Total rotation of joints due to X_3

Joint rotation	*Transverse beam rotation* θ*	*Joint rotation from deflection D*	*Total rotation* $\theta + D$
D_{23}	0.004	–1.025	–1.025
D_{33}	0.0036	1.1511	1.1514
D_{43}	0.00014	–0.1465	–0.1463

*These values are small and can be neglected also.

Part IIIC Correction Analysis Joints 4 and 5

(Apply $X_4 = X_5 = 1$) Apply unit moment at junctions 4 and 5 ($X_4 = X_5 = 1 = 100$ cm.kg/cm run)

(Due to symmetrical loading, there will be no load on plate 5.)

Item 12: **Find beam rotation θ.**

$$\theta_{34} = \frac{200d}{Eh_4^3} = \frac{200\times 140}{2\times 10^5 \times 10^3} = 0.014\times 10^{-2}$$

Slab rotation at joint 4 due to X_4 and X_5

$$\theta_{4,(4/5)} = \left[\frac{400d_4}{Eh_4^3} + \frac{400d_5}{Eh_5^3}\right] + \frac{200d_5}{Eh_5^3} = \frac{(400\times 140)+(400\times 10)+(200\times 10)}{2\times 10^5\times 10^3} = 0.04\times 10^{-2}$$

$$\theta_{5,(4/5)} = \frac{200d_5 + 400d_5 + 400d_6}{2\times 10^5 \times 10^6} = 0.04\times 10^{-2}$$

(These are also small but we will consider them with D values.)

Calculate plate and joint rotations D

Item 5: **Find reactions** $R(X_4 = X_5 = 100$ cm.kg/cm)

$$R_3 = \frac{-100}{d_4 \cos\phi_4} = -0.9322 \text{ kg}$$

$$R_4 = \frac{-100}{d_4\cos\phi_4} + \frac{100}{d_5\cos\phi_5} + \frac{-100}{d_5\cos\phi_5} = 0.9322 \text{ kg}$$

$$R_5 = \frac{-100}{d_5\cos\phi_5} + \frac{100}{d_5\cos\phi_5} + \frac{100}{d_6\cos\phi_6}$$

$$= \frac{100}{d_6\cos\phi_6} = \frac{100}{140\cos 319.58} = 0.9322 \text{ kg}$$

Item 6: **Resolve** R **forces to** P **forces**

$$P_n = -R_{n-1}\left(\frac{\cos\phi_{n-1}}{\sin\alpha_{n-1}}\right) + R_n\left(\frac{\cos\phi_{n+1}}{\sin\alpha_n}\right)$$

Item 9: **Find deflection of plates**

TABLE FOR ITEM 9 Calculation of deflection for X_4 and X_5 = 100 kg.cm/cm

Plot No.	f_{n-1}	f_n	y_n
1	–4.8	9.6	–1.181
2	9.6	–21.0	0.3584
3	–21.0	20	–1.689
4	20	–7	0.321
5	–7.2	–7.2	0

Items 10 and 11: **Find rotation of plates and joints for $X_4 = X_5$ = unity**

TABLE 10B, 11B (FOR ITEMS 10 AND 11) Plate and Joint Rotations (D values)

Plate	*Rotation of Plate*	*Joint Rotation (Slab)*
2	$D_2 = 0.00564$	$D_{24} = D_2 - D_3 = 0.1326$
3	$D_3 = -0.1270$	$D_{34} = D_3 - D_4 = -0.1457$
4	$D_4 = 0.01876$	$D_{44} = D_4 - D_5 = 0.0187$

Item 12 was carried out at the start of Part IIIC.

Item 13: **Total rotation of joints** ($D + \theta$)

TABLE FOR ITEM 13 Total rotation of joints for X_4 and X_5

Joint rotation	*Transverse beam rotation θ*	*Joint rotation from deflection D*	*Total rotation $\theta + D$*
D_{24}	0.00014	0.13264	0.13278
D_{34}	0.00014	–0.14576	–0.14536
D_{44}	0.00014	0.01876	0.01916

Part IV—Solve for X_2, X_3 and X_4

Find values of X_2, X_3, and X_4.

We have to see that total *rotation at each* joint is zero (refer Step 3 of each part).

Joint 2, $X_2D_{22} + X_3D_{23} + X_4D_{24} + D_{20} = 0$

Joint 3, $X_2D_{32} + X_3D_{33} + X_4D_{34} + D_{30} = 0$

Joint 4, $X_2D_{42} + X_3D_{43} + X_4D_{44} + D_{40} = 0$

TABLE 11 Rotation of joints for X values @ 100 cm.kg/cm

	Joint 2	*Joint 3*	*Joint 4*
D_{n0}	–0.6434	–0.7374	+0.0818
$D_{n2}(X_2 = 1)$	1.2345	–1.0233	+0.1330
$D_{n3}(X_3 = 1)$	–1.025	1.1511	–0.1465
$D_{n4}(X_4 = 1)$	+0.1327	–0.1453	0.01916

Write equations for condition for rotation = 0

Find correction moments X_2, X_3, and X_4. (See Table 10, page 235 for D_{20} to D_{40})

Joint 2, $1.2345\ X_2 - 1.025\ X_3 + 0.1327\ X_4 - 0.6434 = 0$

Joint 3, $-1.02336\ X_2 + 1.1511\ X_3 - 0.1453\ X_4 - 0.7374 = 0$

Joint 4, $0.1330\ X_2 - 0.1465\ X_3 + 0.01916\ X_4 + 0.0818 = 0$

Solving these equations, we get values of X in m.kg/cm.

$$X_1 = 4.005 \text{ m.kg/cm}$$
$$X_2 = 4.263 \text{ m.kg/cm}$$
$$X_3 = 0.515 \text{ m.kg/cm}$$

As we have used X_2, X_3, X_4 on 1 m.kg/cm = 100 cm.kg/cm in order to get m.kg/metre. We multiply the result by 100.

It gives,
$$X_2 = 4.005 \times 100 \text{ cm.kg/m}$$
$$X_3 = 4.263 \times 100 \text{ cm.kg/m}$$
$$X_4 = 0.515 \times 100 \text{ cm.kg/m}$$

Part V—Find final longitudinal stresses (in kg/cm^2**) with corrections for design of plates** $X_2 = 4.005$; $X_3 = 4.263$; $X_4 = 0.515$

TABLE 12 Calculation of longitudinal stresses in plates in kg/cm^2

(These stresses are used for determination of longitudinal steel)

Item	*Joints*				
	0	1	2	3	4
1. Uncorrected stresses (Page 219—Table 8)(kg/cm^2)	–1145	+334	–195	–21	+46
2. Correction for X_2 (Page 237—item 8) × 4.005	1041 (259.8 × 4.005)	114 (28.6 × 4.005)	–558	+612	–272
3. Correction for X_3 (Page 240—item 8) × 4.263	198	–391	+703	–668	+290
4. Correction for X_4 (Page 242) × 0.0515	–2.5	4.9	–10.8	10.4	–3.7
Net Stresses	91.5	+61.9	–60.8	–66.6	60.3

Part VI—Find Transverse Moments for Design of Transverse Steel
(These moments are assumed the same all along the span)

TABLE 13 Find transverse moments in m.kg/m span (or cm kg/cm)

Item	*Analysis*	*Joint*	*Plate 1*	*Joint*	*Plate 2*	*Joint*
		0	*Mid-point*	1	*Mid-point*	2
1	Preliminary* (m.kg/m)	0	–1.7	–6.8	+36.0	–44.8
	(For X = 100 cm.kg/cm)					
2	$X_2 = -4.005$**	–	–	–	–200.3	–400.5
3	$X_3 = 4.263$	–	–	–	–	–
4 and 5	$X_4 = 0.515$	–	–	–	–	–
	Total	0	–1.7	–6.8	–158	–445.3

TABLE 13 (CONTD.)

Item	*Plate 3*	*Joint*	*Plate 4*	*Joint*	*Plate 5*	*Joint*
(as above)	*Mid-point*	3	*Mid-point*	4	*Mid-point*	5
1	–32	–32.7	+30.1	–34.5	–27.8	–34.5
2	–200	–	–	–	–	–
3	–213	–426	–213	–	–	–
4 and 5			–25.8	–51.5	–51.5	–51.5
Total	–445	–458.7	–208.9	–86.0	–79.3	–86.0

[Notes:

1. Item 1: See page 216. The mid-point moments in the transverse direction are found by beam analysis of each plate in the transverse direction (width) with the top loading and testing moments on the two ends.

   Example: Calculation of mid-point elements

   (a) With loads on span for middle of plate 2 = 1/2 (sum of moments of joints 1 and 2) + $wL^2/8$. At centre of plate 2 – load 476 kg, span 1.07 m, = 1/2(–68 – 445.3) + 1/8(476 × 1.07 × 1.07) = –158.
2. From X_1 = 100 kg/cm we got X_1 = 4.005. Hence in m kg/m units value is 400.5 at joint 2. Similarly others, midpoint values are one half that at joint as seen from Fig. 16.2.
3. This example shows that the correction effect can be large as the value in preliminary analysis is only +36. We provide steel for the largest moment in the transverse direction.]

REVIEW QUESTIONS

1. What is meant by correction analysis?
2. Explain the terms "plate rotation" and "joint rotation".
3. Total Joint rotation in the Simpson's method has two components, one major and another minor which can be neglected. What are these?
4. What are the methods available for correction analysis in addition to Simpson's method?

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17

EXAMPLE TO ILLUSTRATE COMPLETE ANALYSIS OF FOLDED PLATES

17.1 INTRODUCTION

In this chapter, we examine briefly the full analysis (preliminary and correction) of folded plates, as described in Chapters 15 and 16, by means of an example.

17.2 EXAMPLE OF COMPLETE ANALYSIS OF FOLDED PLATES

A reverse trough (U-shaped) folded plate is 20 m in span and five plates of section as shown in Figure 17.1. Analyze the structure using the manual method. The horizontal plates are 12 cm thick loaded at 395 kg/m^2 and the inclined slabs are 10 cm thick loaded with 300 kg/m^2.

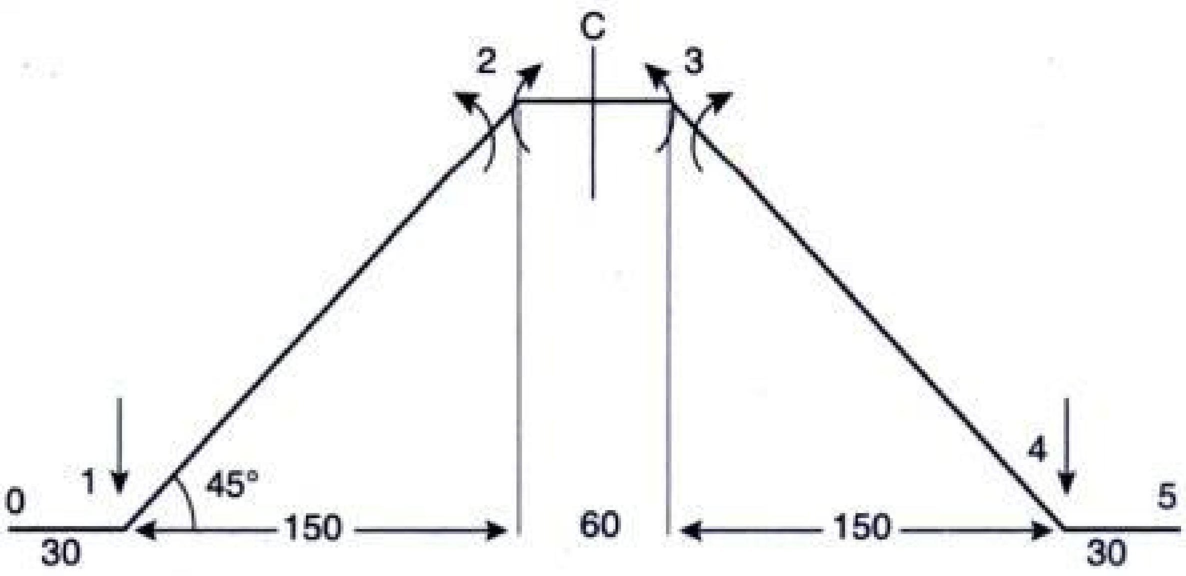

Figure 17.1 Analysis of a folded plate (dimensions are in cm. Load on horizontal plate is 395 kg/m^2 and inclined plate 300 kg/m^2).

TABLE 1 Dimensions of folded plates (Span L = 20 m) (Step 1)

Plate No.	$\frac{\textit{Horizontal width}}{\textit{Sloping}}$ (cm)	*Thickness of plate* (cm)	ϕ_n (degrees)	α_n (degrees)
1	$\frac{30}{30}$	12	0	45
2	$\frac{150}{212.1}$	10	45	45
3	$\frac{60}{60}$	12	0	45
4	$\frac{150}{212.1}$	10	135	–45
5	$\frac{30}{30}$	12	0	–45

ϕ is measured from plate to horizontal and α by extending the plate, both clockwise.

Assume a total UDL of 395 kg/m^2 on horizontal plates and 300 kg/m^2 on inclined plates.

TABLE 2 Geometric properties of plates in cm units (Step 2)

Plates	*Section Area* (cm^2)	*I for transverse analysis* (cm^4)	*For longitudinal analysis*	
			I(cm^4)	Z(cm^3)
1 and 5	360	2.7×10^4	1,800	14,400
2 and 4	2121	795.1×10^4	75,000	833
3	720	21.6×10^4	7,200	14,400

TABLE 3 Load and support moment for transverse analysis

(Horizontal plates @395 kg/m^2 and inclined plates at (in cm kg/meter width) 300 kg/m^2)

We take one metre length along the span (see Table 3. Ex. 15.3 page 215).

Plate No.	*Horizontal Span*	*Load horizontal* (kg/m)	*Support moments* (cm.kg/m along span)
1 and 5	30 cm	395 (cantilever) (3.95 kg/cm)	+1778
2 and 4	150 cm	300 along length	∓7947
3	60 cm	395	∓1185

Example. Calculation of moments in cm kg units M_F of plate 3 = $\dfrac{3.95 \times 60 \times 60}{12}$ = 1185 cm.kg/m along span.

$$M_F \text{ plate } 3 = (3 \times 212.1 \times 150)/12 = 7953 \text{ (cm kg)}$$
$$= 7947 \text{ (approx.)}$$

TABLE 4 Transverse analysis moment distribution (in cm.kg/m span)
[Plates symmetrical about centre line]

	0 — 1	1 —	— 2	2 — 3
	0	1	0.2	0.8
Fixing moment	+1778	−7947	+7947	−1185
Dist. + C.O.		+6169	+3084	
Dist. + C.O.			−1969	−7877
	1778	−1778	+9062	−9062

[Dist. = Distribution and C.O. = Carry Over]
[**Note:** Moment is in cm.kg/m along span.]

TABLE 5 Calculate the *R* forces at supports (kg/m) (Symmetrical about plate 3) (Step 5)

Plate No.	*Description*	R_0	R_1	R_2
1 Reaction	From UDL	0	+118.5	0
2 Reaction	From UDL	0	+318.5	+318.5
2 (Table 4)	*From moment	0	−48.6	+48.6
3 Reaction	From UDL	0	0	+118.5
	Total	0	−388.4	+485.6

$$\text{*Load from moment} = \frac{(9062 - 1778)}{150} = 48.6$$

TABLE 6 Calculation of in-plane *P* forces from forces (See Figure 17.1)

Plate No.	*From Left Support*	*Right Support*	*P Load* (kg/m)
1	0	+388.4 (0 to 1)	388.4 (0 to 1)
2	549.2 (3 to 2)	686.8 (3 to 2)	1235.8 (3 to 2)

(We now work f in kg/cm^2; P above in kg/m)

Calculation of stresses (f values) at ends of plates (L = 2000 cm)

$$M = \frac{wl^2}{8} \text{ and } f = \left(\frac{P}{100}\right) \times \frac{L^2}{8Z} = \frac{5000P}{Z}$$

[**Note:** $P/100$ = load in kg per cm span length as we work in that unit.]

TABLE 7 Calculation of stresses at joints

Plate No.	*Plate Load* (kg/m)	Z (cm^3)	f(kg/cm^2)
1	388.4	1800	± 1078.9
2	1235.8	75000	± 82.8

Example: $Z \text{ of plate } 1 = \frac{bd^2}{6} = 12 \times 30 \times \frac{30}{6} = 1800 \text{ cm}^3$

(1) Area of plate 1 = 30 × 12 = 360 cm^2

(2) Area of plate 2 = 212.1 × 10 = 2121 cm^2

(3) Eq. area of 3 = 2 × (60 × 12) = 1440 cm^2

$$\text{Distribution factor} = \frac{1}{A}$$

$$\frac{1}{360} = 0.00278; \frac{1}{2121} = 0.00047; \frac{1}{1440} = 0.00069$$

$$\text{Distribution factor } F_1 = \frac{0.00278}{0.00325} = 0.855\text{; and } 0.145$$

Similarly factor, $F_2 = 0.4$ to 0.6

TABLE 8 Stress Distribution for compatibility of stresses (Step 8)

(Distribution factor = $1/A$; Carry over = $-1/2$)

	0	1	1	2	2	3
		0.855	0.145	0.4	0.6	
Stresses	−1078	+1078	+82	−82		
Dist. + C.O.	+425	−851	+144	−72		
Dist. + C.O.			−31	+62	−93	
			Continue			
Final values	−639	+200	+200	−94	−94	

End of Primary Analysis

Part II—Calculation of plate and joint rotation of preliminary analysis (Steps 10 + 11)

TABLE 9 Calculation of deflections of plates (Step 9)

Plate No.	kg/cm^2		d_n	y(cm)
	f_{n-1}	f_n		
1	−639	200	30	56.97
2	+200	−94	212.1	−2.83

Step 10 and 11: **Tables 10 and 11—Calculation of plate rotation and joint rotation for preliminary analysis [Eq. (16.2)]**

$$D_n = -\frac{1}{d_n}\left[\frac{y_{n-1}}{\sin \alpha_{n-1}} - y_n(\cot \alpha_{n-1} + \cot \alpha_n) + \frac{y_{n+1}}{\sin \alpha_n}\right] \qquad \text{[Eq. (16.2)]}$$

Joint rotation, $D_n - D_{n+1}$

$$D_2 = -\frac{1}{212.1}\left[\frac{56.97}{-0.707}\right] = 0.3798$$

Joint rotation = $D_2 - D_3 = 0.3798$

TABLE FOR STEPS 10 AND 11 Joint rotations in preliminary analysis

Plate	*Plate Rotation*
2	$D_2 = 0.3798$
3	0

Joint	*Joint Rotation*
2	$D_{20} = 0.3798$

D_{20} = (Joint rotation of joint 2 = 0.3798) [letter 0 represents preliminary analysis]
This has to be corrected by the correction analysis.

[**Note:** We will designate the various steps in correction analysis similar to preliminary analysis as various items. In this example, item 5 of correction analysis will be similar to step 5 of preliminary analysis.]

Part III—Correction Analysis

Because of symmetry, applying unit moment X_2 and $X_3 = 1$ Determination of rotation in unit moment analysis X_2 and $X_3 = 1$ m.kg/cm for each joint [Refer Fig. E17.1]

Item 12: **Calculate beam rotation of beam analysis (θ values) with** $X_2 = 1$ cm.kg/cm

(a) Calculate rotation θ [This may even be neglected]

From Eq. (11.4), rotation θ of joint 2 due to unit load at $X_2 = 1$ and X_3 (*Note:* We apply 1 m.kg/cm or 100 cm.kg/cm for easiness in calculations.)

$$\theta_{2(2,3)} = \frac{4d_n}{E(h_n)^3} + \frac{4d_{n+1}}{E(h_{n+1})^3} + \frac{2d_{n+1}}{E(h_{n+1})^3} \quad \text{[We work in kg and cm units.]}$$

$$= \frac{4\times 212.1}{2\times 10^5 \times (10)^3} + \frac{4\times 60}{2\times 10^5 \times (12)^3} + \frac{2\times 60}{2\times 10^5 \times (12)^3}$$

$$= \frac{1}{2\times 10^5}\left[\frac{4\times 212.4}{1000} + \frac{6\times 60}{(12)^3}\right] = 5.29\times 10^{-4}$$

Rotation for 100 cm.kg/cm = 0.000529 (see Table for Items 10 and 11)

TABLE 12 For θ values for X_3 and X_4 = 100 cm.kg/cm (Part IIIA)

θ values	*Joint 0*	*Joint 1*	*Joint 2*
	0	0	5.29×10^{-4}

(This is negligible and can also be neglected.)

(b) Calculate plate rotation D resulting from deflections

Calculate plate rotations (D values) $X_2 = X_3 = 100$ cm.kg/cm

We go through steps 5 to 11 of the Winter and Pei method which we call items 5 to 11.

[**Note:** Steps of preliminary analysis are designated as items in correction analysis. Thus, the following Item 5 corresponds to step 2 of preliminary analysis.]

Item 5: (Similar to Step 5)—Calculate joint reactions R_{n-1}, R_n and R_{n+1} by Eq. (11.5) for (X = 1 m.kg/cm = 100 cm.kg/cm) [Eq. (15.5)]

Joint 1 reaction = $\dfrac{100}{212 \times \cos 45}$ = 0.667 kg [P effect of other reactions on plates 2 and 3 cancel out.]

TABLE 5A (Item 5)—R forces for X_3 and X_4 (Part IIIB)

Plate No.	*Description*	R_0	R_1	R_4
1	Due to X_2 = 1 and X_3 = 1	0	+0.667	–0.667

Note: Other reactions cancel out.

Item 6: Resolve to P forces
P forces on plate 2 cancel out. P forces act only on plate 1 (see below).

Item 7: Find stresses

TABLE FOR ITEMS 6 AND 7 P forces and f values for X_3 and X_4

Plate	*P forces* (kg)	*Stresses* (kg/cm^2)
1	0.667	±185.8
2	0	0
3	0	0

***Similar to Tables 5 and 6**

Item 8: Make stress distribution

TABLE FOR ITEM 8

	0	1		2	
	0	0.855	0.145	0.4	0.6
Stress	–185.3	+185.3	0.0		
Dist.		–158	26.9		
C.O.	+79.2			–13.45	
Dist.				5.44	–8
C.O.			–2.7		
Dist.		–2.3	+0.4		
Final	–105	+24.6			–8.11

Item 9: Find plate deflection [Eq. (16.1)] from stresses.

TABLE FOR ITEM 9 Calculation of deflection

Plate No.	$f_1 - f_2$	d_n	y
1	129	30	8.784
2	–32.9	212.1	–0.313

Items 10 and 11: Plate rotation and joint rotation. Use Eq. (16.2) and find rotation D from deflections ($D_{n0} = D_n - D_{n+1}$); $D_{20} = D_2 - D_3$ [Eq. (16.3)]

TABLE FOR ITEMS 10 AND 11 Plate and Joint rotation

Plate	*Plate Rotation*	*Joint*	*Joint Rotation*
2	$D_2 = \dfrac{8.744}{212.1(-0.707)}$ $= 0.0585$	2	0.0585

Item 13: Find total joint rotation ($\theta + D$)

TABLE FOR ITEM 12 AND 13 Total Joint Rotation

Joint	θ	D	*Total Joint Rotaion*
2	0.00053	0.0585	0.0595

(We may neglect θ as well.)

Part IV—Solve for unknown X

Rotation (correction analysis + preliminary analysis) = 0

Solve for $X_2 = X_3$. Joint rotation is zero.

$$0.0591X_2 + 0.3798 = 0$$

$$X_2 = -6.50 \text{ m.kg/cm } (650 \text{ cm.kg/cm} = 650 \text{ mm.kg/m span})$$

Part V—Corrected longitudinal stresses in plates (in kg/cm^2**) for reinforcement design**

	0	1	2
Ele. Analysis	−639.6	+200.3	−94.4
Correction ($X_2 = -6.5$)	+682.5	−159.9	+52.0
Final Stresses	+42.9	+40.4	−42.4

Example: Stress at 0 = (−6.5) × (−105) = +682.5 [Refer Table for item 8]

Part VI—Corrected transverse moments (cm.kg/m) **for design of thickness of plates and transverse steel (in** kg.cm/m**)**

(Refer Table 4)

	Joint 0	*Joint 1*	*Joint 2*
Ele. Analysis	0	−1778	−9062
Correction	0	0	−65000
Final	0	−1778	−74062

[**Note:** For detailed tabulation, see Example 16.1.]

18

DESIGN OF REINFORCEMENTS IN FOLDED PLATES AND SUPPORTING DIAPHRAGMS

18.1 INTRODUCTION

In ordinary reinforced concrete slabs, we do not provide any shear reinforcements. In beams we always have to provide at least the minimum shear steel. In folded slabs, the slabs are considered as inclined beams in the longitudinal direction. Therefore, they are subjected to inplane stresses (which some call membrane stresses) and also shear stresses. Shear is very important in folded plates as it is the shear that keeps the joints between the slabs together. The ideal procedure is to find the principal stresses and provide steel. Many recommend this procedure. Alternatively, we can provide steel for shear separately. Thus, in folded plates we have to provide steel for the following items:

1. Transverse steel for the moments in the slab, we got in transverse beam analysis. (The transverse moment got by moment distribution and later corrected by correction analysis to be exact also determines the thickness of the slabs.)
2. Longitudinal steel at the tension side of the slab for the tension in the slab. We have considered the slab as a beam in the longitudinal direction (membrane forces. Minimum specified steel is provided in compression zone also.)
3. Shear steel for the diagonal shear caused by the combined action of the longitudinal stress and the shear.

The first two items above are simple as they are the conventional types and therefore in this chapter, we deal with shear in folded plates and then deal with the others.

References 1 to 4 may be consulted for more detailed information of this subject.

18.2 SHEAR IN FOLDED PLATES

1. Shear at joints: From Chapters 15 and 16 where we examined the integrity of the joints, it is clear that it is the shears at the joints that make the independent plates to act

together as one unit. Thus, the shear between plates at the joints is very important. Let us denote the shear at joint n by the symbol, T_n. Starting from one end of the plate system in the transverse direction, the *shear between joints* can be expressed as follows. Let d be the width of the plate and d its thickness. Figure 16.1 shows the action of shear between the adjacent slabs.

$$T_0 = 0 \text{ (at joint 0 per unit length along span)}$$

$$T_n = T_{n-1} + \frac{A_n}{2}(f_{n-1} + f_n) \quad \text{where } A_n = \text{Sectional area} = h \times d$$

or $$= (\text{Thickness} \times \text{Width}) \text{ of plate} \tag{18.1}$$

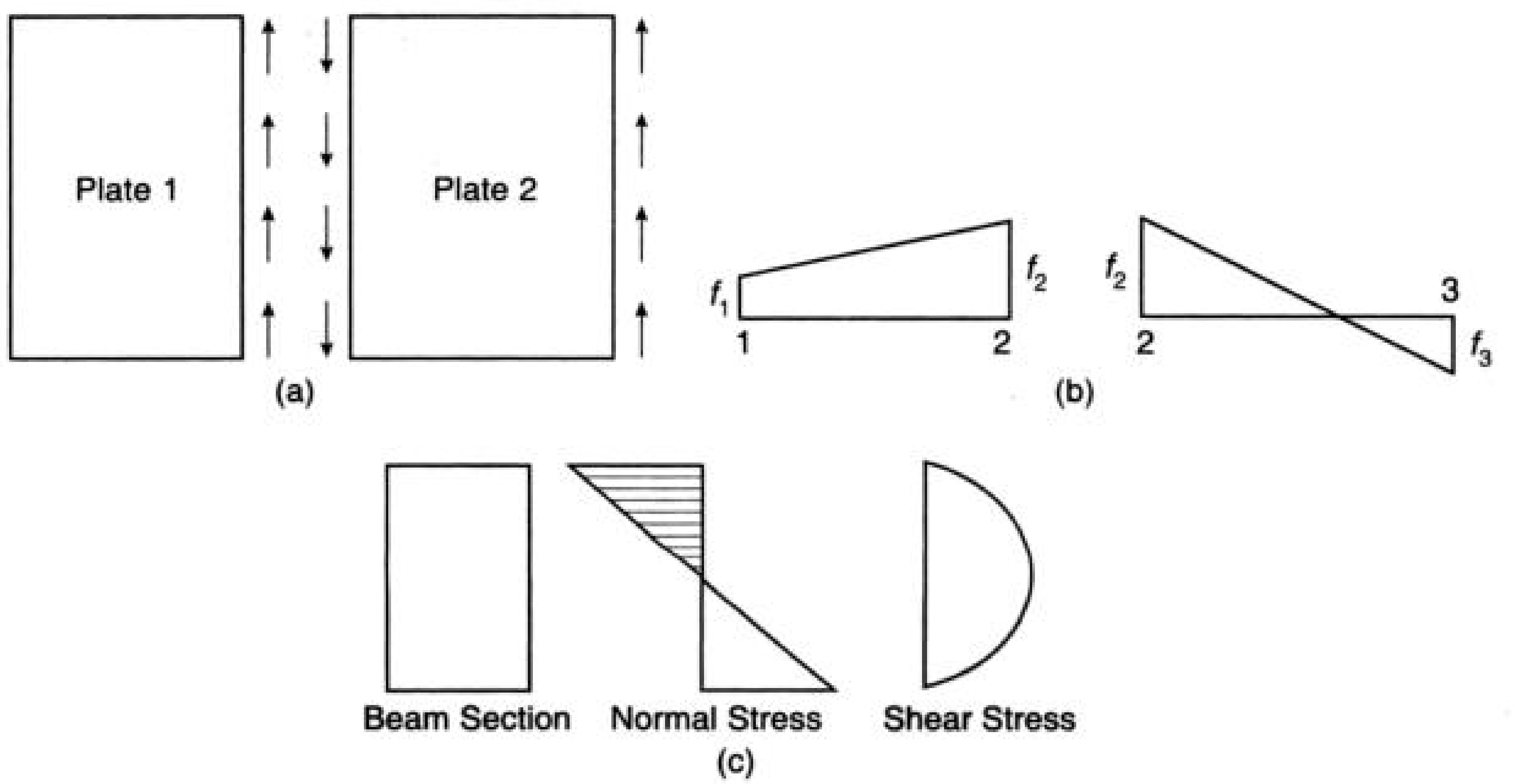

Figure 18.1 Distribution of shears in folded plates (a) Shear between plates at the joint (b) Stress distribution in plates (c) Analogy with shear in simple beams.

At the final joint (end), it will be again zero. It is the magnitude of *shear stresses* at the joint and between the joints that we are interested in. For the analysis of this problem, we make the important assumption that the shear force at the joints has the same type of *longitudinal variation* as the longitudinal moment. We then proceed as follows.

For a UDL bending moment, M at a distance x from support $= \dfrac{wL}{2}x - \dfrac{wx^2}{2}$. It is maximum at $\dfrac{L}{2}$.

$$= \frac{wL^2}{8}\left[\frac{4x}{L} - \frac{4x^2}{L^2}\right] = M_{\max}\frac{4x}{L}\left(1 - \frac{x}{L}\right)$$

Thus, from the above assumption, the tension along the length of the plate at x from support,

$$T = T_{\max}\frac{4x}{L}\left(1 - \frac{x}{L}\right) = T_{\max}\frac{4}{L}\left(x - \frac{x^2}{L}\right) \tag{18.2}$$

These T forces are resultant of the shearing stresses along the length of the plate. Taking h as the thickness of plate,

Let v = shear stress along the length of plate.

Assuming, $T = h\int v\, xdx$, we have $v = \dfrac{1}{h}\dfrac{dT}{dx}$.

Thus, we have the relation that the *shear stress* v is equal to the rate of variation of this edge shear per unit thickness h. Hence, shear stress can be computed from Eq. (18.2) as,

$$v = \frac{1}{h}\left(\frac{\partial T}{\partial x}\right) = \frac{4T_{\max}}{(h \times L)}\left(1 - \frac{2x}{L}\right) \tag{18.3}$$

As this shear can be considered as a cosine variation, this will be maximum at $x = 0$ and $x = L$. The maximum value of shear at the support will be as follows:

Maximum shear at support $x = 0$ $v = \dfrac{4T_{\max}}{(h \times L)}$ (where h = thickness of plate) (18.3a)

[We use this expression for the calculation of maximum shear (Example 18.1).]

The principal tensions provided by v and f (membrane stresses) can be computed from,

$$\sigma_{1.2} = \frac{f}{2} \pm \sqrt{\left(\frac{f}{2}\right)^2 + v^2} \tag{18.4}$$

At supports, f value will be small and the principal tension is diagonal at 45°. In any case, we generally provide nominal diagonal steel (say, 8 mm @ 100 mm to 1/8th span and this is increased to 8 mm @ 200 mm) as shown in Figure 18.4.

Example 18.1 gives the method for the design of folded plates for shear.

It is important that the sectional area provided should be safe in shear. We should provide shear steel if needed. In any case at least the nominal shear steel required must be provided. It will be better if we design the steel so that it takes the full shear without assistance of the concrete in cases where shear excedes allowable concrete stress.

2. Direction of placement of shear steel: In theory, the shear reinforcement should be placed in the direction of the principal stresses near the supports. However, in practice, we place them at 45° as shown in Fig. 18.2. As the joints between plates shears are as in Eq. (18.2), the joint can be in tension or compression and hence the arrangement of reinforcement will be as shown in Figures 18.2 and 18.3 along the direction of the tension force. Each plate can be considered as a beam with given distribution of stress. Thus, *from the analogy of providing bent up bars for shear in beams*, the layout of shear steel in each plate can be determined (i.e. each plate can be considered as a beam and we can use elastic design as given in IS 456).

18.3 DESIGN OF STEEL FOR TRANSVERSE MOMENTS

In our analysis of the folded plate, we correct the transverse moments obtained by preliminary analysis at the joints and middle of the slab by the correction analysis. We

provide steel for the corrected values. Usually, we design for the maximum value of the negative and positive moment *and provide the same steel both on the top and bottom throughout the cross-section.* The folds must be properly detailed to prevent tension cracks. If we design the structure on sine loading, we may decrease the spacing of the transverse steel in the transverse sections as we go towards the support. The layout of transverse steel is shown in Figure 18.3.

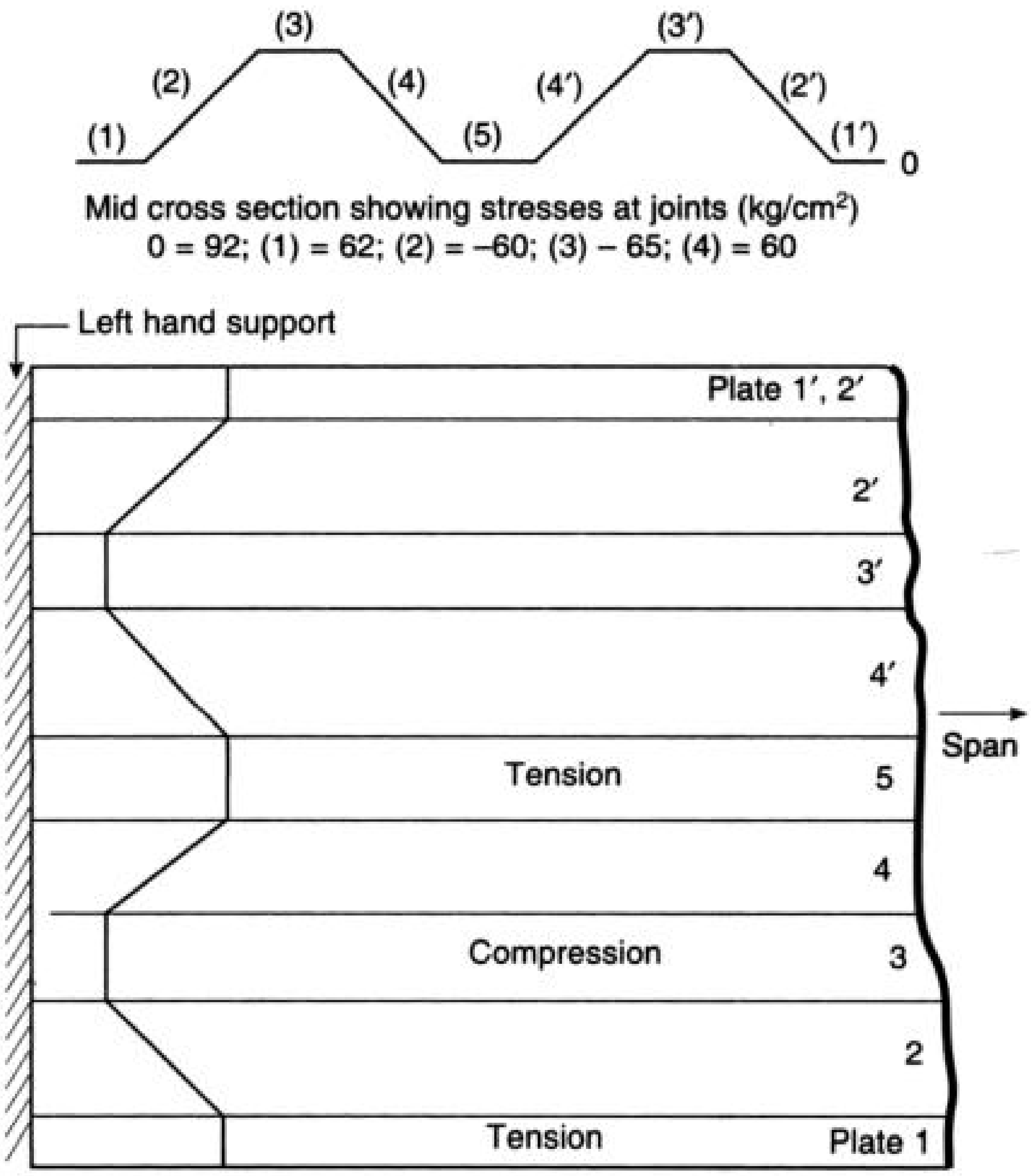

Figure 18.2 Shear reinforcement in folded plates.

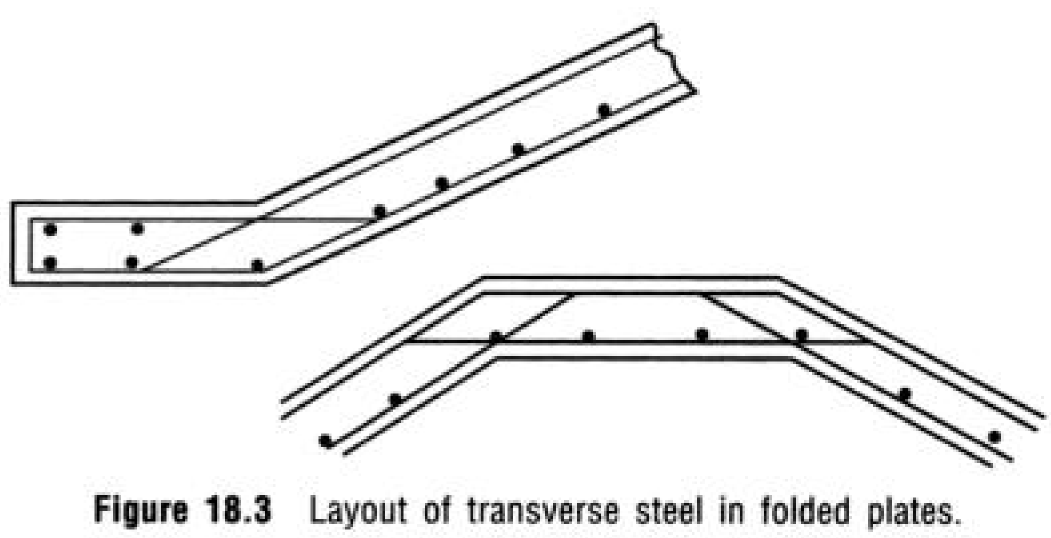

Figure 18.3 Layout of transverse steel in folded plates.

18.4 DESIGN OF LONGITUDINAL STEEL

Each of the various plates when taken as a unit along its width may be fully in tension or fully in compression or partially in tension and partially in compression. We follow the following rules for design:

1. Where the full width of the plate is in tension, we provide steel for the total tension in the plate and distribute the steel throughout the plate.
2. Where part of the width of the plate is partly in tension and partly in compression, we provide for the total tension but place the steel at the tensile edge of the plate (region of high tension) for the entire tension as in the case of steel in ordinary beams. For crack control in any tension region, the steel should not be less than 0.35 percent.
3. Even in the compression zone, we provide some steel not less than the minimum for slabs (usually 8 mm or 10 mm @ 200 mm at least in the compression zone. See also Section 18.6).

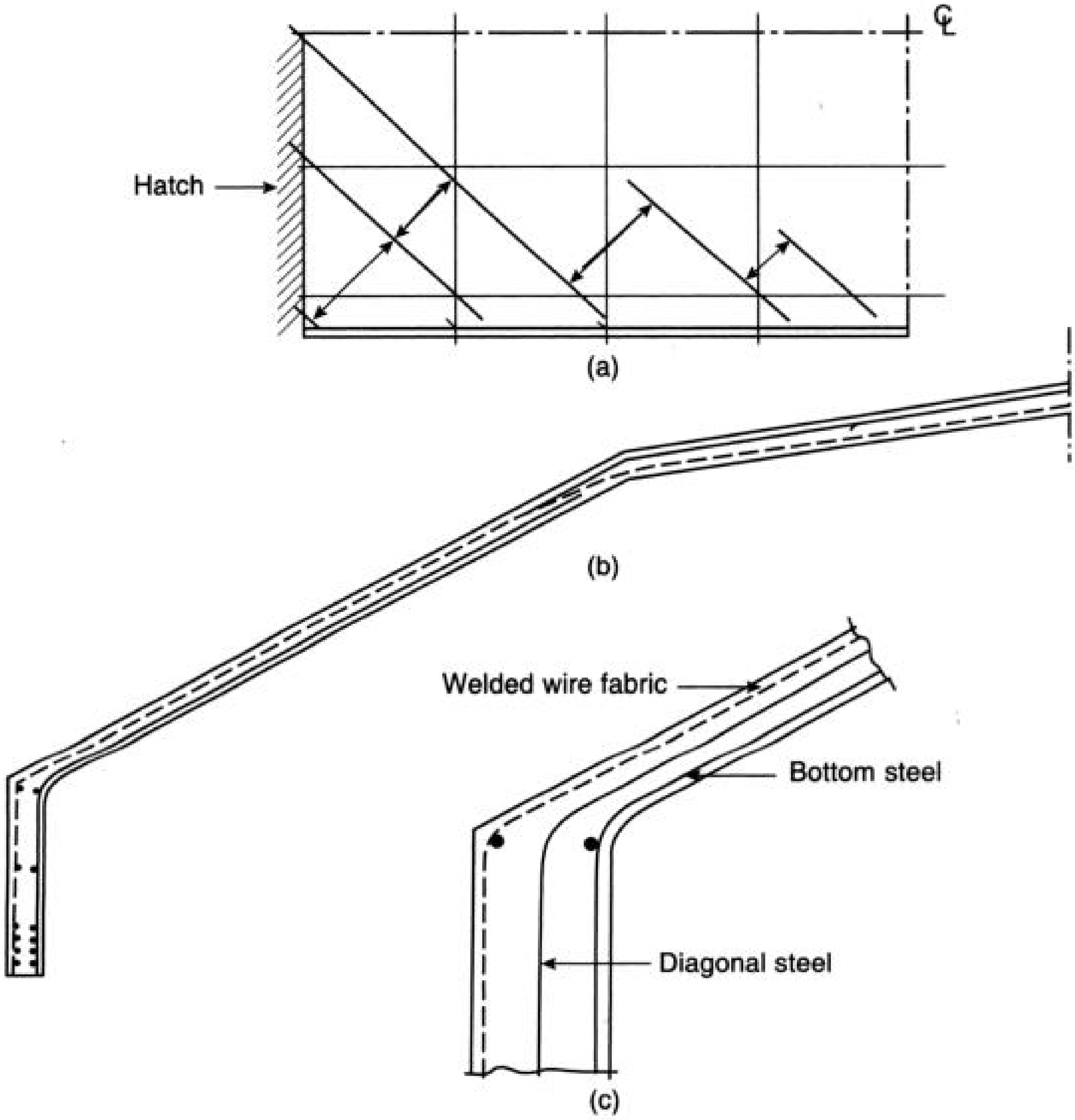

Figure 18.4 Detailing of steel in folded plates for a single barrel cylindrical shell: (a) Placement of shear steel elevation (b) General arrangement of steel at midspan (c) Details at joint 1.

18.5 DESIGN OF DIAPHRAGM

We have assumed that the plates will be supported at their ends in such a way that they can act as beams. Therefore, it is very important that they are supported on proper diaphragms which, in turn, are supported on columns. The loads for which the diaphragms are to be designed are generally taken as follows:

1. The self-weight of the diaphragm which needs no explanation.
2. P forces—The reactions from the plate. This will include the self-weight of plates and line loads on the plates. It equals $P \times (L/2)$. These P forces along the width of plates are resolved into vertical and horizontal forces.

We will examine the second aspect in more detail. Example 18.4 illustrates the procedure.

18.6 DETAILING OF STEEL

IS 2210-1988, Clause 12.2 gives the following recommendations for detailing of reinforcements for folded plates. Three types of steel are provided:

1. *Transverse reinforcement*. It should resist the transverse moment and should follow the cross-section of the folded plate.
2. *Longitudinal reinforcement*. It should take up the longitudinal stresses.
3. *Diagonal steel* should be provided for shear.

The section of the plate at its junction with the transverse (support) should be *checked for shear stress caused by edge shear*.

General. Nominal steel for the compression zone should be at least 8 mm at 200 mm (in important cases 10 mm may be used). The minimum steel should conform to IS 456-2000. So, the spacing of main steel should not be more than 300 mm or 3 times the depth. Nominal steel should not be spaced more than five times the thickness of slab or 450 mm. The area of unreinforced panel shall in no case exceed 15 times the square of thickness.

The minimum diameter of the bar should be 8 mm and the maximum diameter should not be more than 1/4 the thickness of shell or 16 mm, whichever is smaller.

The maximum amount of steel allowed per metre width is generally specified as $(f_c''/f_s) \times 100$ percent. Thus with f_c'' as 70 kg/cm^2 and f_s = 1400 kg/cm^2, it will work out to 5 percent.

SUMMARY

The reinforcements in the plate consist mainly of the transverse steel, longitudinal steel and shear steel. The supports should be designed as a beam for carrying the forces transmitted from the plate to the supports. In all these cases, the IS recommendations should be followed.

EXAMPLE 18.1 [Checking folded plate for shear]

The f values for the folded plate in Example 15.3 continued as Example 16.1 are shown in Figure 18.2. Design the folded plate for shear.

Reference	*Step*	*Calculations*
Fig. E18.1	1.	*Find shear between joints*
See Table 12		Take stresses at joints 0 to 4 = 92, 62, –60, –65, 60 kg/cm^2
Page 244		$T_n = T_{n-1} + \frac{A_n}{2}(f_{n-1} + f_n)$
		Joint D, $T_d = 0$
		Joint 1, $T_1 = +\frac{20 \times 10}{2}(92 + 62) = +15400 \text{ kg}$
		Joint 2, $T_2 = +15400 + \frac{1400}{2}(62 - 60) = +16800 \text{ kg}$
		Joint 3, $T_3 = +16800 + \frac{400}{2}(-60 - 65) = -8200 \text{ kg}$
		Joint 4, $T_4 = -8200 + \frac{1400}{2}(-65 + 60) = -11700 \text{ kg}$
		Joint 5, $T_5 = -11700 + \frac{400}{2}(60 + 60) = +12300 \text{ kg}$
	2.	*Find the point of maximum shear between plates and its value*
		Stress at 1 = +62. Stress at 2 = –60
		Distance of zero stress from joint 1.
		$x = \frac{62 \times 140}{(62 + 60)} = 71 \text{ cm}$ from joint 1
		$T_{max} = 15400 + \frac{62}{2} \times 71 \times 10 = 37410 \text{ kg/m}$
		This is the total tension from the edge to point of zero tension.
Eq. (18.3a)	3.	*Calculate shear stress*
		$v = \frac{4T_{max}}{h \times L} = \frac{4 \times 37410}{10 \times 1800} = 8.31 \text{ kg/cm}^2$
IS415		In N/mm^2 = 0.83 Mmm^2 (Requires designed steel).
Table 23		(We design as in cylindrical shells. Steel is placed diagonally (with direction as with diagonal steel in beams) for principal stresses taken equal to shear stresses as direct stress is small.
		[Take max shear at support in plate 2 = 0.83. It varies as cosine function = 0.83 cos $\pi x/L$.]

Reference	*Step*	*Calculations*
		Design for max shear per meter length (4000 mm) and thickness of plate = 100 mm $V = 0.83 \times 1000 \times 100 = 83{,}000$ N
IS 456		Using 415 steel $f_s = 230$
Table 22		Area of steel As = $\frac{83{,}000}{230} = 360$ mm^2
$f_s = 140$		12 mm @ 200 mm gives 565 mm^2 provide 12 mm at 20 cm diagonally
		[Notes: 1. As given in Sec 18.2 numerical shear steel should be at least 8 mm at 20 cm. 2. Draw this steel on a diagram and then calculate steel required along the length not covered by assuming the shear to maximum in the plate along the plate 2 with $x = 71$ from joint 1 as in step 2 and varies as a cosine function. 3. Study derivation of shear design in beams [5].]

EXAMPLE 18.2 [Design of steel in the transverse direction in folded plates]

In the transverse beam analysis of a folded plate, *the maximum moment obtained is* about 460 kgcm/cm run. Determine the steel required if the thickness of the plate adopted is 100 mm.

Reference	*Step*	*Calculations*
	1.	*Find effective depth* $d = 100 - 15 - 5 = 80$ mm
	2.	*Check depth (thickness) required* Fe 415 steel M20 concrete $M = k\sigma_c bd^2 = 0.917\ bd^2$ $M = 460$ cm kg/cm width $= 460 \times 100$ N/10 mm $b = 10$ mm; $d = \sqrt{\frac{46000}{0.917 \times 10}} = 70.8 < 80$ mm Available $d = 80$ mm (thickness 100 mm)
	3.	*Find area of steel required using elastic design (As it is a roof and crack control is needed, we will use elastic design. We may also use limit state design.)* $A_s = \frac{M}{f_s \cdot j^d} = \frac{46{,}000}{230 \times 0.9 \times 80} = 2.77$ mm^2 for $B = 10$ mm Spacing of 10 mm rods = $\frac{10 \times 78.5}{2.77} = 283$ mm. Adopt 10 mm @ 275 mm spacing top and bottom.

EXAMPLE 18.3 [Design of longitudinal steel in folded plates]

Figure 18.1 shows the distribution of stresses in a folded plate. Design the longitudinal steel required.

Reference	*Step*	*Calculations*
		Design of Plate 1 (Fully in tension)
	1.	*Find total tension* (*Length* = 20 cm) *and thickness* (t = 10 cm)
		$$T = \left[\frac{92+62}{2}\right] \times 10 \times 20 = 15400 \text{ kg} \quad \text{[plate breadth = 20 cm]}$$
	2.	*Find area of steel required (Elastic Design)*
		$$A_s = \frac{154000(N)}{230} = 669 \text{ mm}^2$$
		4 rods of 16 mm gives 804 mm^2
		These rods are placed equidistant (with cover at the ends) in plate 1.
		Design of Plate 2 (Partly in tension and partly in compression)
	1.	*Find point of zero stress*
		Let it be at x from joint 1.
		$$\frac{x}{62} = \frac{140}{122} \quad \text{or} \quad x = 71 \text{ cm}$$
		$$\text{Total tension} = \frac{62}{2} \times 71 \times 10 = 22010 \text{ kg}$$
		$$A_s = \frac{220100(N)}{230} = 957 \text{ mm}^2$$
		5 rods of 16 mm gives 1005 cm^2
		This steel is provided in the tension zone.
		(We may check composite tension in the region also.)
	2.	*Check stress at compression zone*
		Max stress in compression = 60 kg/cm^2 (6 N/mm^2)
		(For M_{20} concrete, f_c in varying compression can be up to 7 N/mm^2. Hence safe.)
	3.	*Provide minimum steel in compression zone*
		Min. steel = 0.12%
		$$\text{Total steel} = \frac{(140-71)\times 10 \times 0.12}{100} = 0.828 \text{ cm}^2 = 83 \text{ mm}^2$$
		Length of compression zone = 69 cm
		Provide 5 nos. 8 mm rods (giving 201 mm^2) equally spaced.

EXAMPLE 18.4 [Design of diaphragms (supports)]

The diaphragm must be designed for self-weight +*P* forces in the plates.

Calculate the *P* forces acting on the diaphragms from the folded slabs given in Example 15.3.

[**Note:** The *P* forces due to preliminary analysis is given in Table 6 of Example 15.3 (Chapter 15). Values of X_1 to X_3 are given in the end of part IV, Example 16 in Chapter 16.]

The following tables give the value of *P* forces in the folded plate of Example 15.3. (In preliminary analysis, we have *P* in kg/m.)

TABLE 1 Correction analysis *X* and *P* values in kg/m along span for X = 100 kg.cm/cm or 1 m.kg/cm

Value of X	Values of P for unit X = 1 m.kg/cm			
	Plate 1	Plate 2	Plate 3	Plate 4
X_1 = 4.0	1.11	3.86	–7.07	3.89
X_2 = 4.26		–3.90	+7.05	–3.887
X_3 = 0.05			–1.103	0

To convert to kg/m run, we multiply above values by 100.

From the above, we get the values of *P* for total analysis.

TABLE 2 *P* forces in plates in kg/per metre run along span

No.	Analysis	Plate 1	Plate 2	Plate 3	Plate 4
1	Preliminary	–322	+998	–116	–908
2	For X_2 = 4.0	+440	1552	–2828	1556
3	For X_3 = 4.260	–	–1655	3006	–1656
4	For X_4 = 0.0515	–	–	–571	–
	Net value	+118	+895	–509	–1008

These *P* forces can be resolved into vertical and horizontal loads on the diaphragm.

For a half length span of 9 m, the forces along the plates will be as follows from net values:

Plate 1 = 118 × 9 = 1062 kg = 1.1*t*

Plate 2 = 895 × 9 = 8055 kg = 8.1*t*

Plate 3 = 509 × 9 = 458 kg = 4.5*t*

Plate 4 = –1008 × 9 = –9072 kg = –9.02*t*

These forces are transferred to the diaphragm and the diaphragm is designed for these loads.

REVIEW QUESTIONS

1. State the principles used in detailing of transverse steel and longitudinal steel in folded plates.
2. Usually shear steel is placed along the diagonal lines up to 1/8th span from the supports. Give the principles in their design.

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2. *For translational shells like hypar, the following formula has been proposed* [2]:
   (a) The formula for critical stress is as follows:

$$\sigma_{cr} = \frac{Ete}{ab\sqrt{3}} \tag{19.2}$$

   where a and b are the sides, e is the height, and t is the thickness of shell.
   (b) Another formula for *critical load in* kg/m^2 *per unit area* is as follow [2]:

$$q_{cr} = \frac{2Et^2e^2}{a^2b^2\sqrt{3}} \tag{19.3}$$

   As regards the value of E to be used, it is to be the long-term value and not the same as in static tests. It has been observed that in reinforced concrete shells as buckling sets in, the value of E decreases greatly. Thus, if we use the standard value of $E = 5000\sqrt{f_{ck}}$ (N/mm^2), *a safety factor of at least four is recommended.*

19.4.1 IS-2210-1988 Clause 9 Recommendations for Checking the Elastic Stability or Buckling of Shells and Folded Plates

1. Allowable compression stress in cylindrical shells and folded plates: The permissible compressive stress f_{ac} is as follows:

$$f_{ac} = \frac{0.25 f_{ck}}{1 + \left(\dfrac{f_{ck}}{f_{cr}}\right)} \tag{19.4}$$

where f_{ck} = Characteristic 28 day strength

f_{cr} = Critical buckling stress to be calculated as specified in IS 2210.

The value of f_{cr} is to depend on the type of shell whether they are short or long (i.e. whether one edge disturbance affects the other or not) depending on the A_{as} Jakobsen's classification parameters ρ and k given below.

$$\rho = 8\sqrt{\frac{(12\pi^4R^6)}{(L^4t^2)}}$$

$$k = \frac{\pi^2R^2}{L^2\rho^2}$$

where R = radius of shell and t = thickness of shell and L = span.

If ρ is less than 7 and k less than 0.12 (are Type I long shells)

If ρ exceeds 10 and k exceeds 0.15 (are Type II short shells)

In these long shells (Type I shells), the longitudinal stresses are critical. In short shells (Type II shells), the transverse stresses are critical. The critical buckling stress is as follows.

Case (1). For Type I shells, $f_{cr} = 0.20\ E_c t/R$ (19.5)

Case (2). For Type II shells—In this case, the transverse stresses tend to be critical from the point of view of buckling and the critical buckling stress f_{cr} shall be determined as follows.

(1) For shells with $L < 2.3\sqrt{dR}$

$$f_{cr} = E_c\left[3.4\left(\frac{d}{L}\right)^2 + 0.025\left(\frac{L}{R}\right)\right] \tag{19.6a}$$

(2) For shells with $L > 2.3\sqrt{dR}$

$$f_{cr} = E_c\left[\frac{0.89\dfrac{d}{L}\sqrt{\dfrac{d}{R}}}{1-1.18\sqrt{\dfrac{d.R}{L}}}\right] \tag{19.6b}$$

where E_c = Modulus of elasticity of concrete which may be taken as = $\frac{E_s}{280} \times 3f_{ck}$

E_s = Modulus of elasticity of steel

d = Thickness of shell

R = Radius of curvature

Case (3). Shells with ρ values between 7 and 10 and κ between 0.12 and 0.15 are relatively infrequent. For such shells, formulae given in the above cases shall apply depending upon whether longitudinal or transverse stresses are critical from considerations of elastic stability.

The value of modulus of elasticity of concrete to be used in the above formulae for calculating the buckling stresses should be the value for long term modulus, including the effect of creep also.

[**Note:** It is interesting to note that in Case (2), for f_{cr}, the length L is also important.]

2. Doubly curved shells: Shells of double curvature are more resistant than shells of single curvature.

Taking the *permissible buckling load* per unit area of surface to be P_{per}.

$$P_{\text{crit}} = \frac{KEt^2}{R_1R_2} \text{ taking } K = 0.1$$

$$P_{\text{per}} = \frac{0.1E_ct^2}{R_1R_2} \tag{19.7a}$$

$$P_{\text{per}} = \frac{0.1E_ct^2}{R^2} \text{ for spherical shells} \tag{19.7b}$$

Actual tests on shells with parabolic directrix gave values of K as low as 0.06. With ellipsoid shells, K was found to be as low as 0.5.

3. Folded plates: For folded plates, it can be considered similar to the corresponding cylindrical shells or as a slab.

4. For hyperbolic paraboloid shells: Fischer recommends the following formula [2], Eq. 19.2

$$f_{cr} = \frac{Ete}{\sqrt{3}(a \times b)}$$

where a and b are the length and breadth and e is the rise of the shell.

19.5 BUCKLING STRENGTH OF SUPPORTING MEMBERS

The total shells structure consists of not only the shell proper but also the supporting members like the transverse frame of cylindrical shells, the edge beams of hypar shells, etc. These members in compression should be also checked for buckling instability [3].

SUMMARY

In this chapter, we examined the maximum compression to be allowed in different types of shells so that buckling does not take place. As the theoretical analysis is generally based on elastic behaviour and the real behaviour is inelastic, we must adopt a high factor of safety of at least 4 that should be used on the value of the buckling stress we calculate in theory.

EXAMPLE 19.1 [Buckling strength of spherical shells]

A reinforced concrete spherical shell is of 50 m in diameter and has thickness of 6 cm. Find the maximum permissible compressive stress for the design of this shell.

Reference	*Step*	*Calculations*
Eq. (19.1)	1.	*Formulae to be used* Zoelly's formula, $\sigma_{cr} = \dfrac{Et}{R\sqrt{3}}$ Assume M25 concrete, $E = 25 \times 10^4$ kg/cm^2
	2.	*Find critical stress* $\sigma_c = \dfrac{25 \times 10^4 \times 6}{5000 \times \sqrt{3}} = 173 \text{ kg/cm}^2$
	3.	*Find allowable stress* Allowing a factor of safety of 4 $\sigma_{max} = \dfrac{173}{4} = 43.2$ kg/cm^2 (= 4.3 N/mm^2)

EXAMPLE 19.2 [Buckling strength of hypar shells]

A hyperbolic paraboloid has its length = width as 7 m ($a = b$). Its rise is $a/5 = 1.4$ m and its thickness is 7 cm. Find the allowable compression to be safe against buckling. Assume M25 concrete, $E = 25 \times 10^4$ kg/cm^2.

Reference	*Step*	*Calculations*
Eq. (19.2)	1.	*Formula for buckling strength of hypar shells* $$\sigma_{cr} = \frac{Ete}{ab\sqrt{3}}$$
	2.	*Calculate σ_{cr}* $$\sigma_{cr} = \frac{25 \times 10^4 \times 7 \times 140}{700 \times 700 \times \sqrt{3}} = 288 \text{ kg/cm}^2$$
	3.	*Calculate allowable compression* Assume F.S. = 4; $\sigma_c = \dfrac{288}{4} = 72$ kg/cm^2 (7.2 N/mm^2)

REVIEW QUESTIONS

1. Explain the following:
   (a) Slenderness ratio
   (b) Brazier effect
   (c) Zeolly's formula
   (d) Critical stress and critical load
2. What is the effect of Gauss curvature on buckling strength of shells?
3. Which stresses are important for buckling in the following cylindrical shells:
   (a) Long shells and
   (b) Short shells?
4. Write down the formula for buckling stress of the following:
   (a) Spherical shells
   (b) Cylindrical shells
   (c) Hyperbolic paraboloids
   (d) Elliptic paraboloids
   (e) Folded plates.

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20

DESIGN OF PYRAMID ROOFS

20.1 INTRODUCTION

A pyramid roof consists of plane triangle surfaces bounded on each side along its height by sloping slabs, giving a common vertex as shown in Figure 20.1. It can be a triangular pyramid with three sides, a square pyramid with four sides, a hexagonal pyramid with six sides or an octagonal pyramid with eight sides, ultimately evolving into a cone. In this chapter, we examine only square pyramid, which is more popular than others.

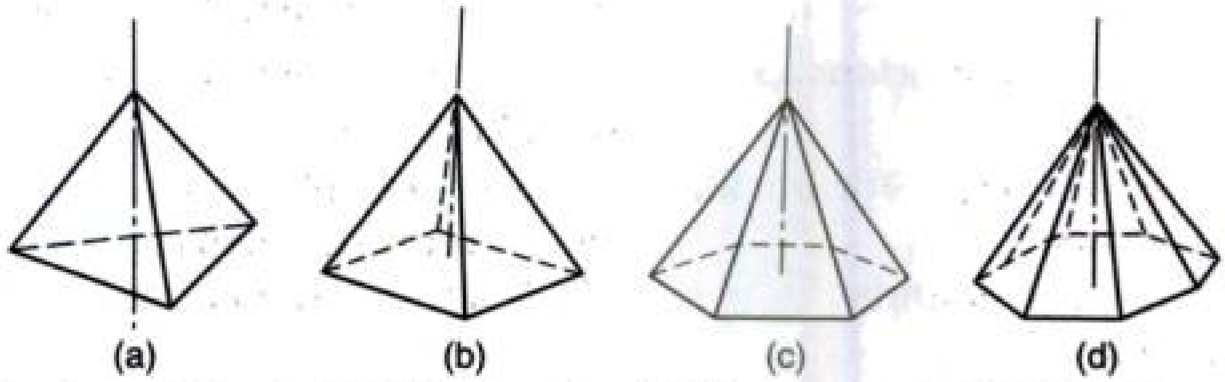

Figure 20.1 Examples of pyramidal roofs: (a) With three sides, (b) With four sides, (c) With five sides, and (d) With six sides.

We have seen that in the case of a dome or a cone, the surfaces are curved and stability is retained by membrane forces (direct tension or compression). In the case of a square pyramid, stability is maintained by bending in addition to direct thrust in two directions at right angles. The pyramid itself may be supported as,

Case (1) Base supported on a wall

Case (2) Base supported by columns at the corners only

The main parts of the pyramid are (i) the sloping surfaces and (ii) *the base beams* or bottom ribs which act like the ring beam in a circular dome. The following elementary method of design is based on Reference 1. We will consider the two case of (a) fully supported base and (b) supported only on columns at corners.

20.2 CASE (1)—ANALYSIS OF A PYRAMIDAL ROOF ON A SQUARE BASE SIMPLY SUPPORTED ALONG LOWER EDGE

We will illustrate the principles of analysis of these roofs with pyramidal roof on a square base. (Proper modification can be made for rectangular pyramids also.) The forces acting on the triangular surface are the following as shown in Figure 20.2:

Effect A: The meridinal thrust N_ϕ along the triangular surface (similar to the meridinal thrust in conical shells).

Effect B: The secondary horizontal thrust N_θ (similar to the hoop compression in cones).

Effect C and D: The N_ϕ value at the base ring beam on resolution at base will produce horizontal and vertical components. The vertical component will be taken by the supporting system and the horizontal component will produce two effects. As the base is not circular instead of ring tension, the horizontal thrust will produce tension (effect C) and bending (effect D) in the bottom ring beam. Therefore, the lower edge should preferably be provided with a rib.

Effect E: The component of the loading normal to the surface of the pyramid will produce bending in the triangular slab as if supported at the edges.

We will examine these five effects (C to E) one by one in detail in the following sections.

20.2.1 Effect (A)—Meridinal Force (N_ϕ)

If the pyramid weighs 'w' per unit area, referring to Figure 20.2, we will consider forces on line ab at a depth y from vertex. (α, as in the case of a cone, is angle of face with the vertical.)

Weight above line $ab = \frac{1}{2}w(2y\tan\alpha)\left(\frac{y}{\cos\alpha}\right)$ where y = depth from apex

Component along slab (N_ϕ) along unit length of ab,

$$N_\phi = \frac{\frac{1}{2}w(2y\tan\alpha)\left(\frac{y}{\cos\alpha}\right)\times\left(\frac{1}{\cos\alpha}\right)}{(2y\tan\alpha)} = \frac{1}{2}w\left(\frac{y}{\cos^2\alpha}\right) \tag{20.1}$$

[**Note:** This is the same expression we have for conical shells, Eq. (4.12) (see Section 20.4). Maximum value is obtained when y = the vertical height of the pyramid.]

20.2.2 Effect (B)—Secondary Horizontal Thrust (N_θ)

For N_θ is similar to hoop compression in conical shells supported at the base. Considering Figure 20.2, we find that there has to be a net inward thrust which has to be resisted by two horizontal reactions (on the adjacent planes). The *compression* per unit length induced can be found as follows:

$$\text{Increase in length of side} = dS = \frac{dy}{\cos\alpha}$$

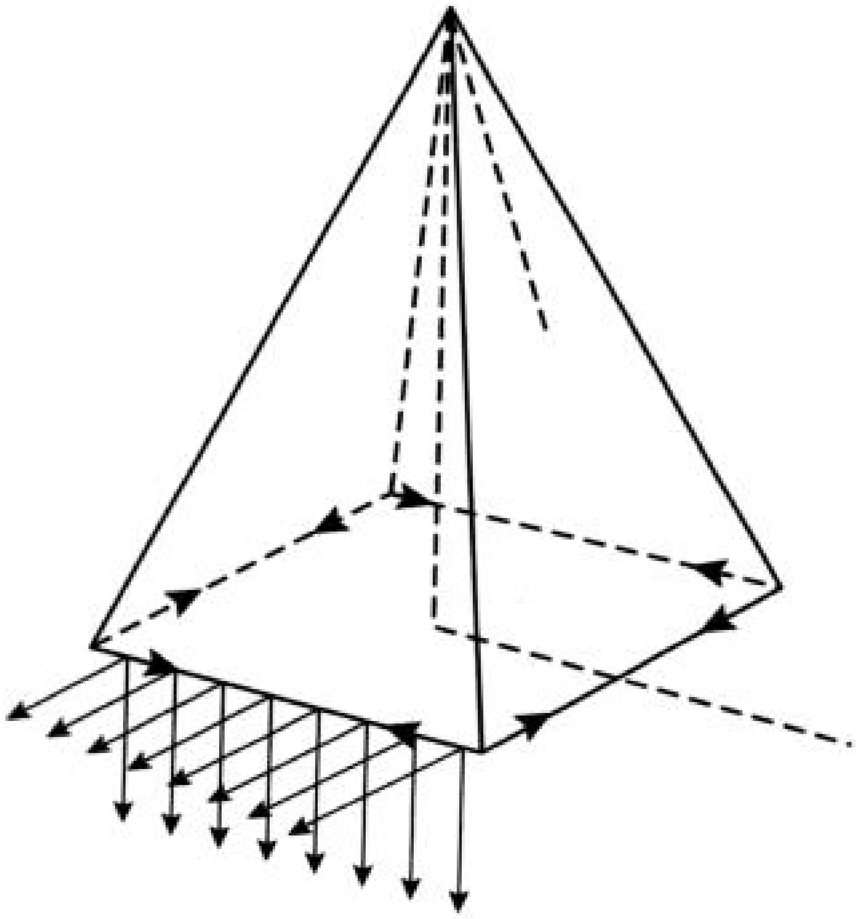

Figure 20.3 Effect C (tension along lower edge).

Let S = base length; $h = \dfrac{S}{2}\tan\alpha$

$$\text{Maximum value of } N_\phi \text{ at base} = \frac{wh}{2\cos^2\alpha} = \frac{w}{2}\left(\frac{S}{2\tan\alpha}\right)\times\frac{1}{\cos^2\alpha} \quad \text{[Eq. (20.1)]}$$

$$\text{Maximum value of } H \text{ per unit length} = \frac{wS}{4\cos\alpha}$$

$$\text{Total value of } H \text{ for length } S = \frac{wS^2}{4}$$

This value has to be resisted by the edge beam on the other two sides.

$$\text{Tension on both sides} = \frac{wS^2}{8}\sec\alpha \text{ (where, } S = \text{base length)} \tag{20.3}$$

(The bottom rib has to take this tension.)

20.2.4 Effect (D)—Bending Moment on the Bottom Rib due to Horizontal Thrust

The horizontal bending due to thrust H per unit length can be taken as shown in Figure 20.3 as follows.

Assuming thrust H, the maximum bending moments ab, the centre can be taken as $HS/24$ and at the corner as $HS/12$. Hence,

$$\text{Max +ve BM in span} = \frac{HS}{24} = \left(\frac{wS}{4}\sec\alpha\right)\left(\frac{S}{24}\right) = \frac{wS^2}{96}\sec\alpha \tag{20.4a}$$

$$\text{Max –ve BM at corner} = \frac{HS}{12} = \frac{wS^2}{48}\sec\alpha \tag{20.4b}$$

20.2.5 Effect (E)—Bending of Slab from Load Normal to It

Effect (E) is the bending of the triangular face due to component of the weight normal to the triangular area. This will be calculated using the standard method for triangular slabs, namely based on the inscribed circle as shown in Figure 20.4.

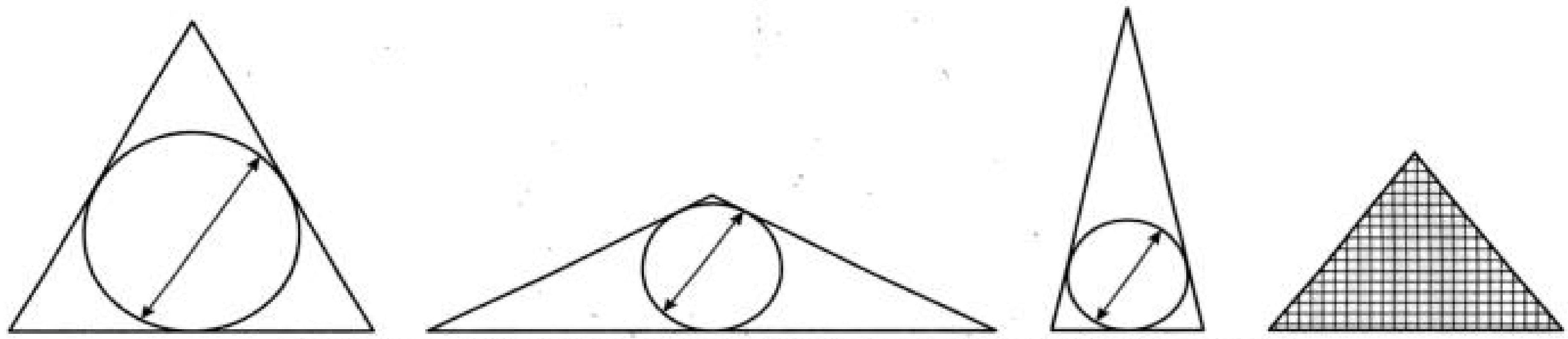

Figure 20.4 Effect E (bending of slabs due to loads normal to the surface).

(a) *For an equilateral triangle*

In the horizontal direction, $M = (w \sin \alpha) \times \left(\dfrac{d^2}{16}\right)$ (20.5a)

In the sloping direction, $M = (w \sin \alpha) \times \left(\dfrac{d^2}{18}\right)$ (20.5b)

(b) *For a triangle with obtuse angle at vertex*

Practically, all load is carried in sloping direction,

$$M = w \sin a \times \left(\frac{d^2}{10}\right) \tag{20.5c}$$

Provide nominal distribution of steel (and at least 20% of main steel) is provided.

(c) *For a triangle with acute angle at vertex*

Load mainly carried in the horizontal direction,

$$M = w \sin a \times \left(\frac{d^2}{12}\right) \tag{20.5d}$$

Normal distribution of steel (not less than 20% of main steel) is provided in the sloping direction.

20.3 CASE (2)—PYRAMID SUPPORTED ON COLUMNS

When the pyramid is supported only by four corner columns (Fig. 20.5), we have to provide beams under the pyramid to take care of the full weight of the pyramid. The beams are then supported at their ends by columns. As the depth varies along the length, we may assume a beam of average depth and design the beam as fixed at the two ends.

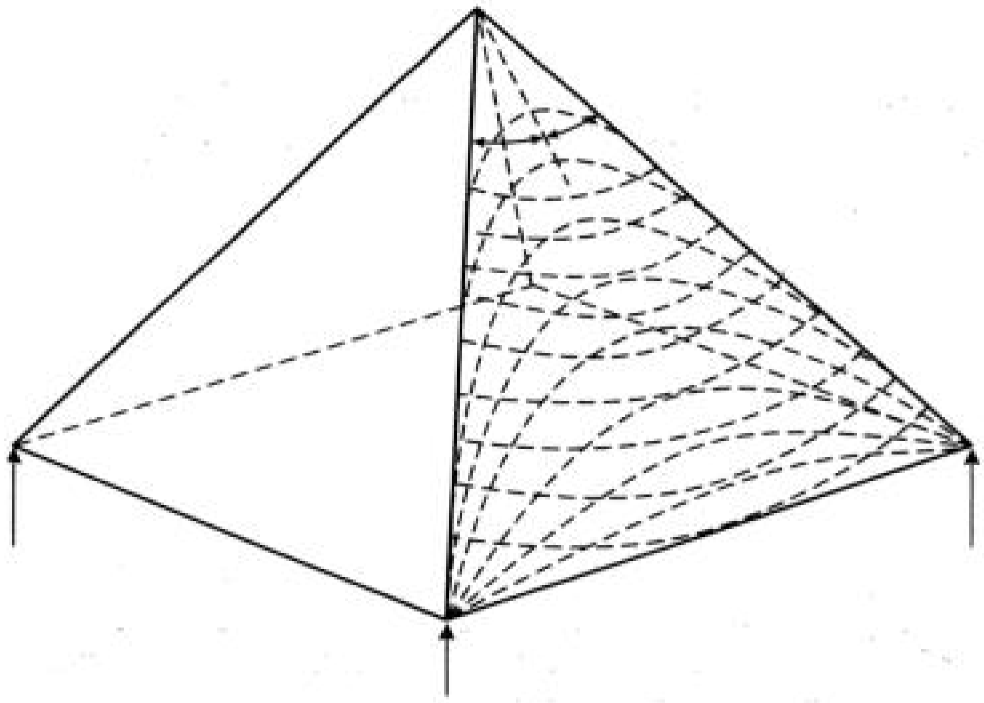

Figure 20.5 Pyramidal roof on columns.

20.3.1 Shear Along the Ridges When Supported on Columns

Supporting the pyramid at its corners only will produce shear along the ridges of the pyramid. The shear along the ridges AV and BV is given by,

$$2R \cos \beta = N_\phi(S)$$

where N_ϕ is the value per unit length. Hence,

$$R = \frac{(N_\phi \times S)}{(2 \cos \beta)} \tag{20.6}$$

where S = length of the base of the triangle and β is the half angle as shown in Figure 20.5.

SUMMARY

This chapter gave one of the methods of design of a square pyramidal roof. The same principles can be applied to rectangular pyramids also.

Taking z as depth from the apex, we get the equation for N_ϕ and N_θ for shells in the form of a sphere, cone and pyramid as follows:

Type of structure	N_ϕ	N_θ
Spherical shell	$\dfrac{wR}{(1+\cos\phi)}$	$wR \cos \phi - N_\phi$*
Conical shell	$\dfrac{wz}{2\cos^2\phi}$	$wz \tan^2\phi$
Pyramid	$\dfrac{wz}{2\cos^2\phi}$	$wz \tan^2\phi$

*In a sphere the hoop stress can turn into a tensile force.

This is a good similarity with the cone and the pyramid where N_ϕ and N_θ are in compression.

EXAMPLE 20.1

Calculate the forces acting on a square pyramidal roof of base 9 m × 9 m and height of apex 4.8 m. Assume the total weight acting on the roof of the slab is 5 kN/m^2 and that the roof is supported on the corners.

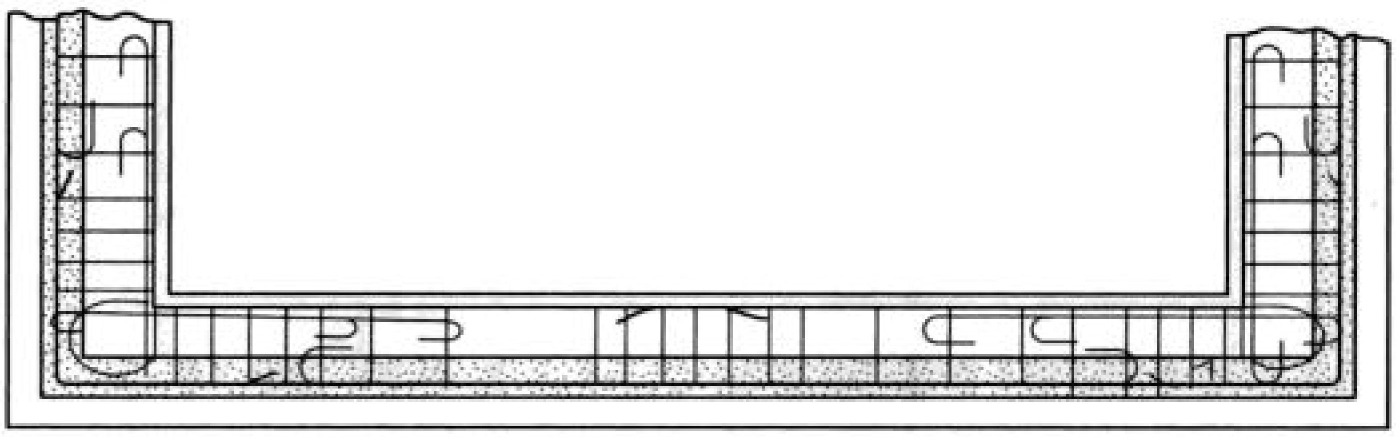

Figure E20.1 Plan of base rib.

Reference	*Step*	*Calculations*
	1.	*Calculate dimensions*
		Side S = 9 m
		Height h = 4.8 m
		(a) Slant length $= \sqrt{(4.5)^2 + (4.8)} = 6.625$ m
		(b) $\tan^{-1} \alpha \left(\dfrac{4.5}{4.8}\right) = 0.938;\ \alpha = 43.2°$
		(c) $\tan^{-1} \beta = \dfrac{4.5}{6.625} = 0.938;\ \beta = 34.2°$
		(d) z of C.G. $= \dfrac{2}{3} \times 6.625 \times \cos 43.2° = 3.22$ m along sloping faces
		We will find magnitudes of "*effects A, B, C, D and E*" as follows.
Eq. (20.1)	2.	*Calculate Effect (A)—Meridinal thrust N_ϕ*
		Max. $N_\phi = \dfrac{wy}{2\cos^2 \alpha} = \dfrac{5 \times 4.8}{2 \times (0.728)^2} = 22.64$ kN/m
		N_ϕ at C.G. level $= \dfrac{(22.64 \times 3.22)}{4.8} = 15.2$ kN/m
Eq. (20.2)	3.	*Calculate Effect (B)—Secondary horizontal thrust N_θ*
		Max. $N_\theta = wz \tan^2\alpha = 5 \times 4.8 \times (0.938)^2 = 21.1$ kN/m
		N_θ at C.G. level $= \dfrac{(21.2 \times 3.22)}{4.8} = 14.2$ kN/m
Eq. (20.3)	4.	*Calculate Effect (C)—Tension along lower edge*
		$T = \dfrac{wS^2}{8} \sec \alpha = \dfrac{5 \times 9^2 \times 1.37}{8} = 69.4$ kN

Reference	Step	Calculations
Eq. (20.4a)	5.	*Calculate Effect (D)—Bending moment due to thrust on rib*
Eq. (20.4b)		Max. BM in span $= \dfrac{wS^2 \sec\alpha}{96} = \dfrac{5 \times 9^2 \times 1.37}{96} = 52.07$ kN.m
		Max. BM at corners $= \dfrac{wS^2 \sec\alpha}{48} = 104.0$ kN.m
Eq. (20.5)	6.	*Calculation of Effect (E)—Bending of slab normal to triangular slab supported on edge*
		Diameter of inscribed circle isosceles triangle sloping length = 6.6 m and base 9 m.
		Bottom angle = 55.8°. Half angle = 27.9°
		Radius = 4.5 × tan 27.9° = 2.38 m
		Diameter d = 2 × 2.38 = 4.76 m
		BM in sloping direction $= \dfrac{w \sin\alpha(d^2)}{18}$
		$= \dfrac{5 \times \sin 45.2 \times (4.76)^2}{18} = 4.5$ kN.m/m
		BM in horizontal direction $= \dfrac{w \sin\alpha(d^2)}{16} = 5.0$ kN.m/m
Step 3	7.	*Design of slab along the slope at C.G.*
Step 6		Design for compression and moment
		N_ϕ at C.G. = 15.2 kN/m (compression)
		BM = 4.5 kN.m/m (Moment)
Step 2	8.	*Design of slab across the slope*
Step 6		Secondary thrust N_θ = 14.2 kN/m (compression)
		BM = 5.0 kN.m/m (moment)
Step 4	9.	*Design of lower edge*
Step 5		Tension = 96.4 kN (tension)
		BM in span = 52.0 kN.m (tension outside)
		BM at support = 104.0 kN.m (tension inside)
	10.	*Design of slab*
		Let the slab be 100 mm in thickness. Design steel for the above forces.
	11.	*Design of base rib*
		Make base rib 250 mm deep and 400 mm broad to take care of the bending. As there is tension on this base rib, we will design for bending on steel beam theory (i.e. concrete does not act) and provide separate steel to take tension.

REVIEW QUESTIONS

1. Compare the values of N_ϕ and N_θ for a sphere, a cone and a pyramid and make your comments.
2. Indicate the difference in design of a square pyramidal roof supported on wall and such a roof supported only by four columns at the corners.

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Appendix A

A SHORT HISTORY OF MASONRY DOMES

A.1 INTRODUCTION

From prehistoric times and even today, in the construction of most buildings, we use two distinct structural elements. First, we use a *covering element* like stone, wood, leaves, roofing sheets or even a concrete slab nowadays to cover the space. Secondly, the covering material, in turn, is supported by *supporting elements* like lintels (or beams) or posts (columns). Most buildings of the Egyptians, the Babylonians, the Greeks and even many of our own buildings consist of these two distinct structural elements, the covering element and the supporting element. Exceptions are the pyramidal structures like the temples of South India.

It was only during the time of the Roman Civilization that spectacular innovations took place in building construction. It was they who developed the masonry domes, which, like modern shell structures, can carry out the dual functions of covering the space as well as supporting by themselves. This Appendix gives a short account of the development of the masonry domes. In our study, we will also examine the construction of a few of the old well-known domes built with masonry construction (which, unlike modern reinforced concrete construction, cannot take any tension forces).

A.2 DEVELOPMENT OF THE DOME

The circular arch is the forerunner of domes. It was the Romans who developed the arch. The arch would have been developed from corbelling that was used from ages for entrances as shown in Figure A.1. The Romans liked the circular shape, which is easy to layout. They used circular arch extensively not only in buildings but also in bridges, aqueducts and other structures.

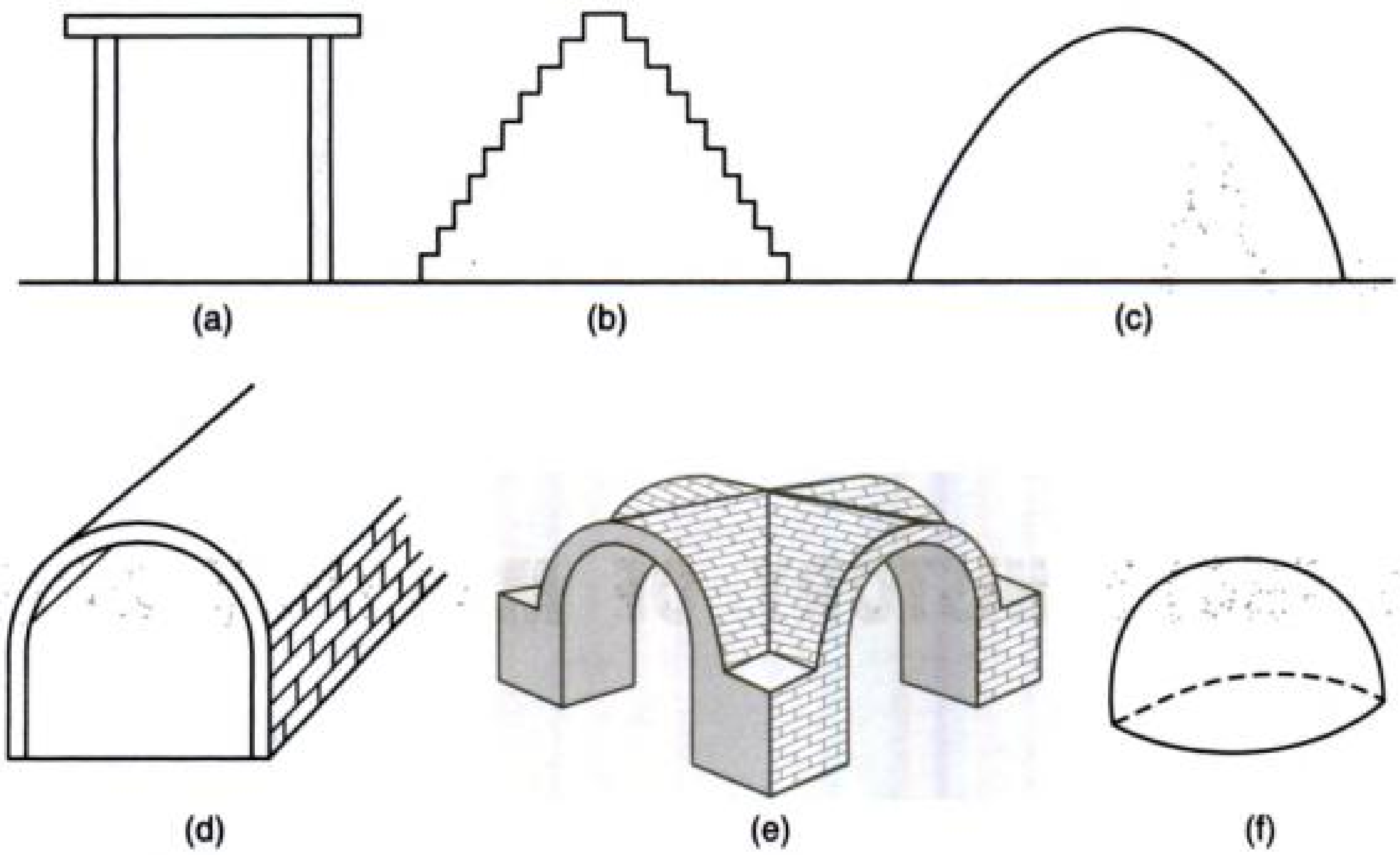

Figure A.1 Evolution of dome: (a) Lintel, (b) Corbel, (c) Arch, (d) Vault, (e) Cross vault, and (f) Dome.

It was quite natural that the arch, a linear structure to be developed into a vault, a space structure as shown in Figure A.1. The Romans found that two vaults in perpendicular directions formed an interesting intersection which can cover a large square area supported by only corner columns. The construction of arches all around the favourite circle of the Romans produced a circular dome.

But the dome produced many problems. Even the arch is always active. As the Arab proverb goes, "The arch never sleeps" [1]. Unless it is fully semicircular, it always exerts horizontal thrust, which tries to push away its own supports. When the arch becomes a dome, it has two problems. First, if the dome is not semicircular but only segmental, *the horizontal thrust acts all around the structure.* Secondly, the stresses in a dome are more complex than those of an arch. The dome exerts not only meridinal compression but also hoop stresses, which turns into tension in the surface below about $\phi = 52°$ about the central axis. Masonry and even Roman Concrete that the Romans developed (see Section 1.11) cannot withstand this tension.

It is interesting to study how ingeniously the ancients solved these twin problems of horizontal thrust and hoop tension in the masonry domes they built. The following methods they invented are worthy of special mention [1].

A.2.1 Roman Method

The Romans used the following three methods to solve the problems of thrust and tension in domes, as shown in Figure A.2:

1. Provision of massive abutment to withstand the large horizontal thrust.
2. Reduction of the weight of the upper part of the dome by making it thinner and providing an opening (called the "eye").
3. By providing *ribs in* the domes [Figure A.2 and Figure A.4).

A.2.2 Byzantine Method (Lobster Pot Method)

The Byzantine method consisted of constructing the dome with *ribbed arches* (like the polygonal dome described in Chapter 13) as shown in Figure A.3 in the construction of St. Sophia in Istanbul. The builders divided the dome into sectors and miniature segmental domes spans only between rib to rib. It is also interesting to note the use of pendentive as a transition from a square to a circle. The introduction of ties at the base as described in Section 4.4.2 was evolved during this period. [The place by Zantine was later named constantinople and is now known as Istanbul.]

A.2.3 Baroque Method

In the Baroque method, the tension on the dome was resisted by tying a chain around the dome at the potential bursting point. The use of iron chain in the construction of St. Paul's Church in London is shown in Figure A.5. (This corresponds to 17th to 18th century.)

A.3 GOTHIC ARCHITECTURE

It is interesting to observe in passing that the Gothic architecture with pointed arches used in many churches is of later origin. It revived the dual system of having division between covering and supporting members. The arch ribs between the opposite pillars are extremely rigid and the covering material is built to be supported by these arches.

A.4 DESCRIPTIONS OF SOME FAMOUS MASONRY DOMES

In this section, we examine briefly the construction of some of the old masonry domes.

A.4.1 Pantheon Dome in Rome (AD 120–124) [2]

The Pantheon dome (Figure 4.2) has an inner diameter of 142.5 feet (43.5 m) as shown in Figure A.2. The rib construction, the reduction of section (and hence the weight) in the upper parts, and the opening at top by the provision of an "eye" are worthy of note.

A.4.2 St. Sophia Dome, Istanbul (AD 532–537)

In St. Sophia dome (Figure A.3), the distribution of tension and also the problem of horizontal thrust in domes was solved by the "Lobster Pot" construction [1]. The surface of the dome was divided into 40 ribs running from lowest circumference to the apex. Thus, the domes are spans only for short distance of rib to rib, thus obviating the tension. (This division into arcs has been adopted in most of the domes built with masonry before the introduction of reinforced concrete, which became popular only after World War I.)

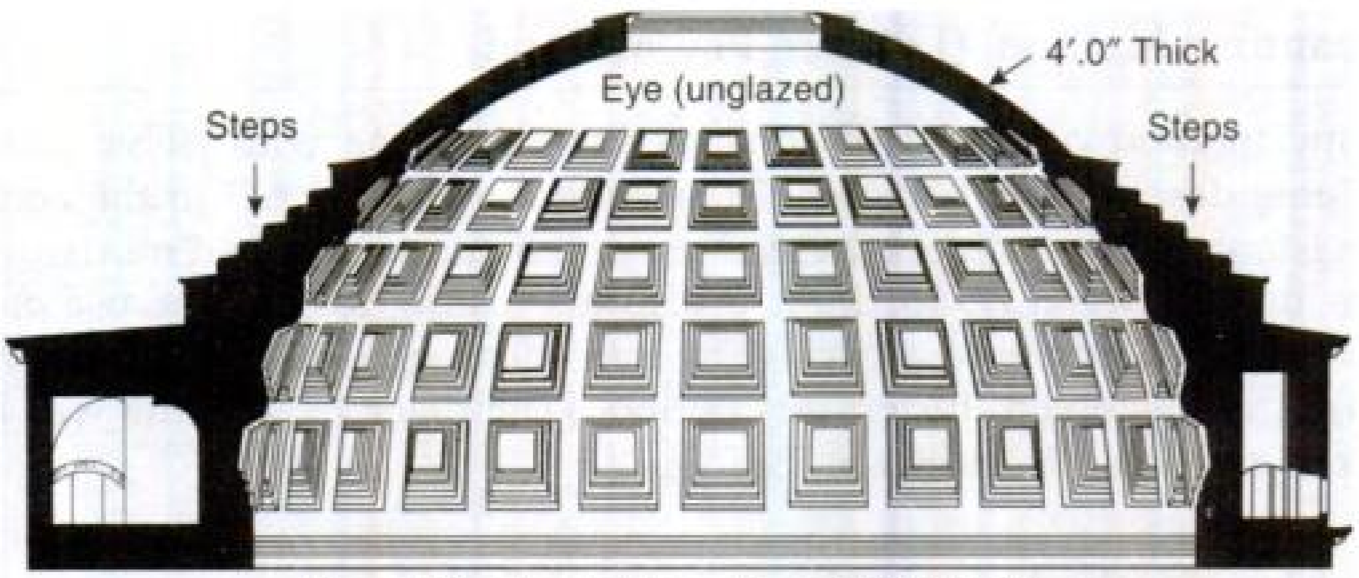

Figure A.2 The Pantheon, Rome (120–124 AD).

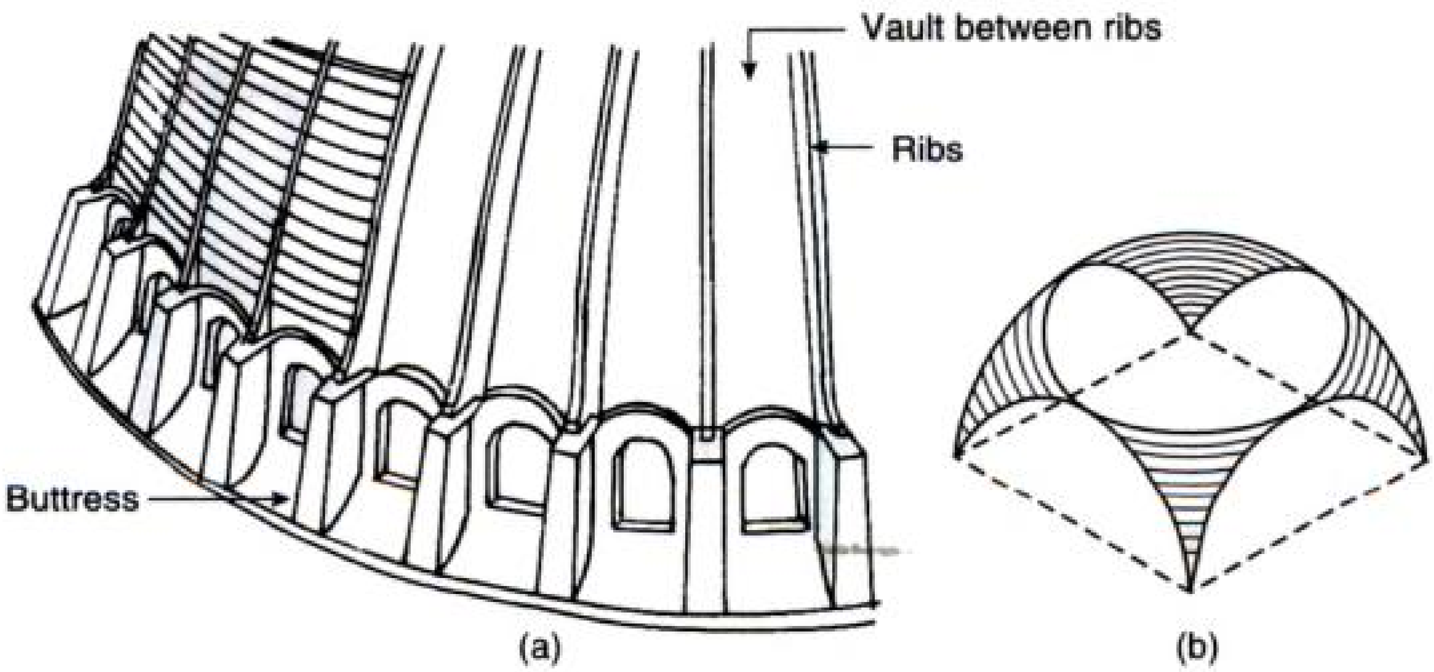

Figure A.3 St. Sophia (532–537 AD). Typical Byzantine dome construction: (a) Dome construction and (b) Pendentive is the transition from square to circle (pendentive dome).

A.4.3 Dome of Florence Cathedral (AD 1420–1434)

Florence Cathedral (Figure A.4) is an outstanding structure of the Renaissance. It is reported to have been designed by Filloppo Brunelleschi who studied Roman remains before he took up the work. The dome is pointed in form to conform to other parts of the church. It has eight main ribs and another sixteen intermediate ribs supporting panels of the brickwork with horizontal joints. It is reported to have been built without much centering.

Its unique feature was the introduction of a hoop of timber secured with iron at the junctions at the base binding them together, thus obviating the need for buttresses to withstand the thrust of the arches. This tie bar, as pointed out in Section A.2.2, was a Byzantine invention [2].

A.4.4 Dome of Taj Mahal Hotel in Mumbai

The dome of the Taj Mahal hotel in Mumbai as shown in Figure A.5, built before the introduction of reinforced concrete in India, is of masonry and is similar in appearance to the Florence dome.

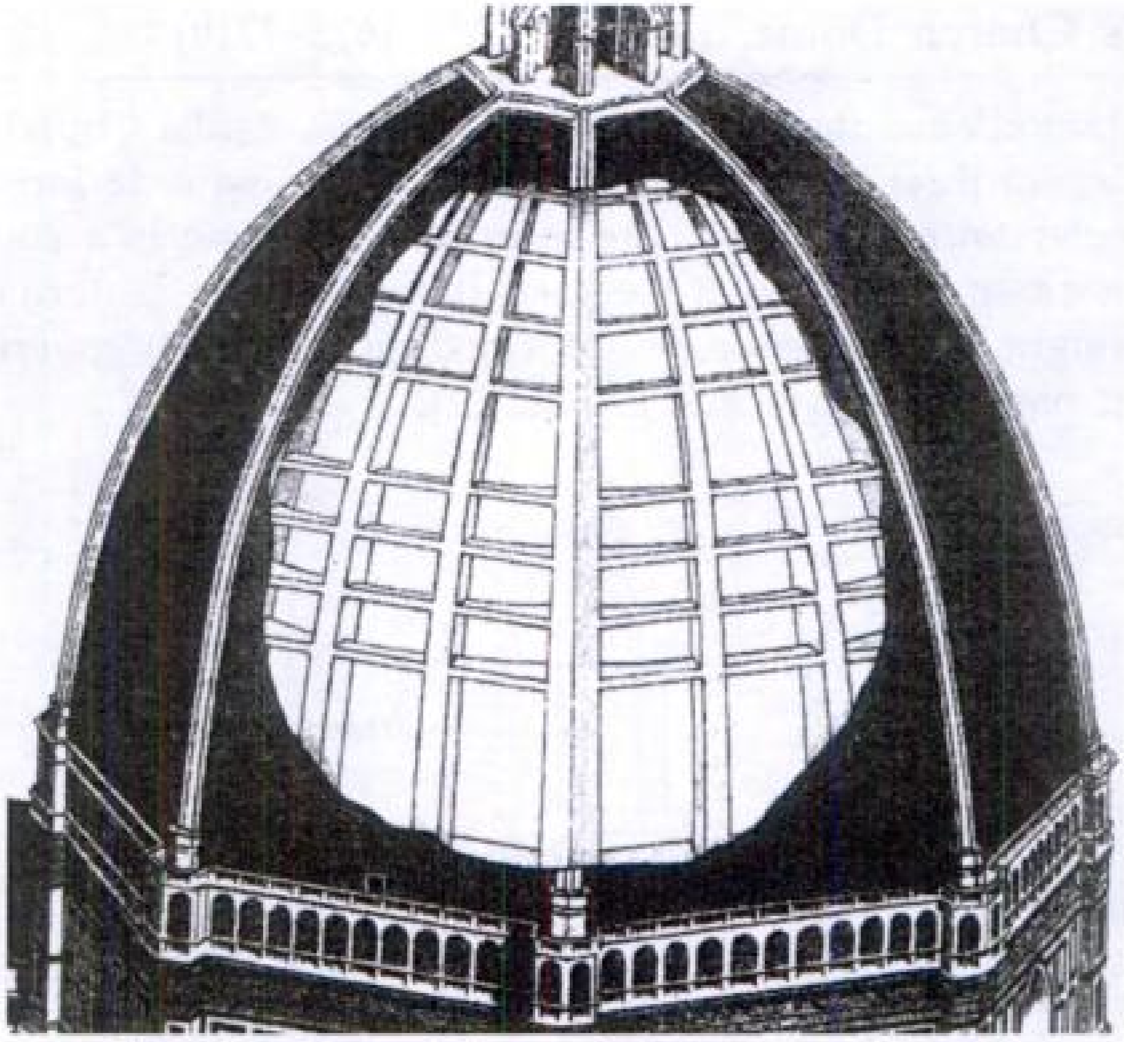

Figure A.4 Dome of Florence Cathedral, Italy (1420–1434).

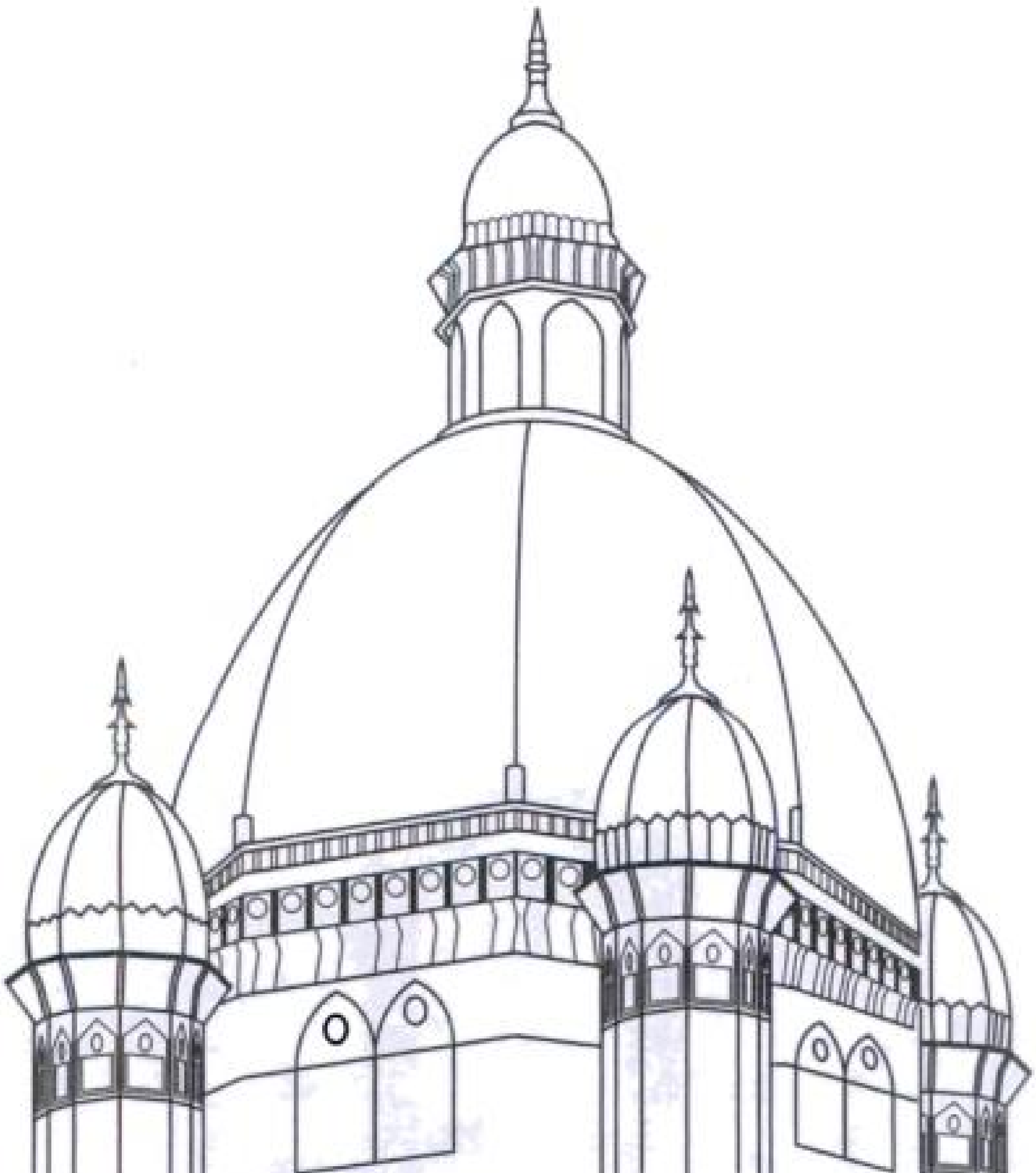

Figure A.5 Dome of Taj Mahal Hotel, Mumbai.

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Appendix B

FUNICULAR SHELLS

B.1 INTRODUCTION—CATENARY SHELLS—FUNICULAR SHELLS

The dictionary meaning of the word "funicular" is, "of a rope or its tension". The shape of a freely hanging rope is a *catenary*. Hence, cylindrical shells of single curvature formed with catenary curves as the directix and a straight line as a generator are called a *catenary cylindrical shells*, in the same way as the circle forms the circular cylindrical shell.

Shells of double curvature with catenary shape in both axes are called *funicular shells*. These are space structures. It is easy to imagine how they can be formed. If a fabric is stretched over a rigid mould of required plan and is loaded with a uniform layer of concrete or plaster of paris, and allowed to take its own shape, we get a funicular shape when the concrete or plaster hardens. The fabric carries the load in pure tension. If we invert it to form a roof, it will be under pure compression under its own weight. Funicular shells work on this principle. Catenary cylindrical shells and funicular shells are shown in Figures B.1 and B.2.

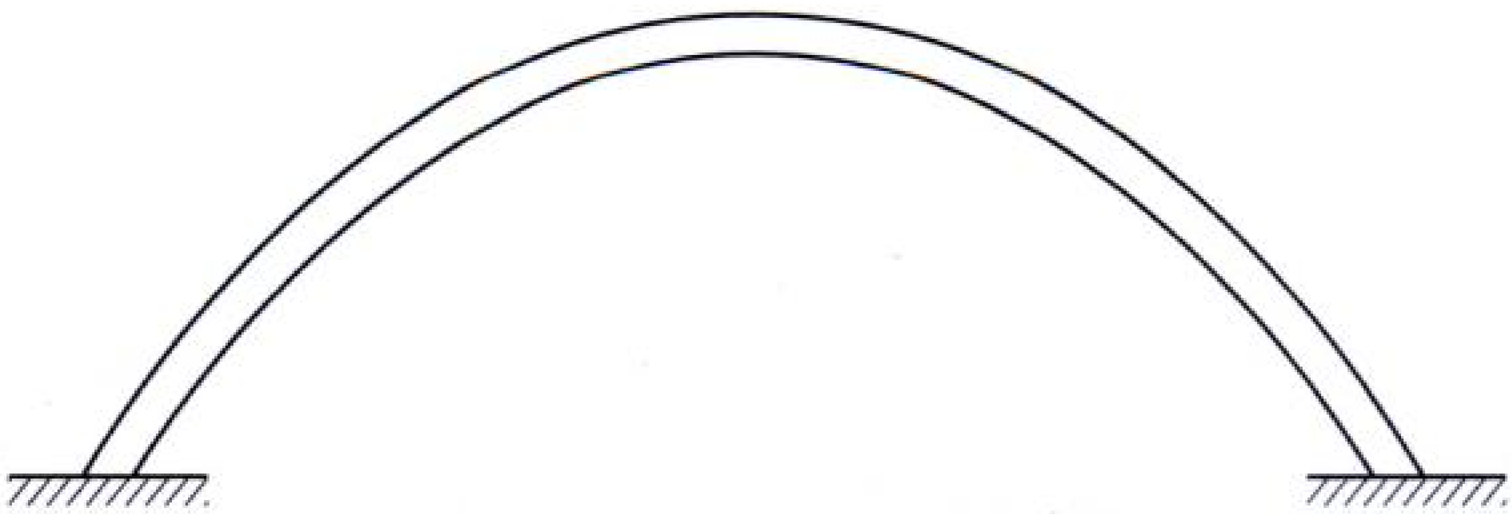


Figure B.1 Catenary shell of single curvature.

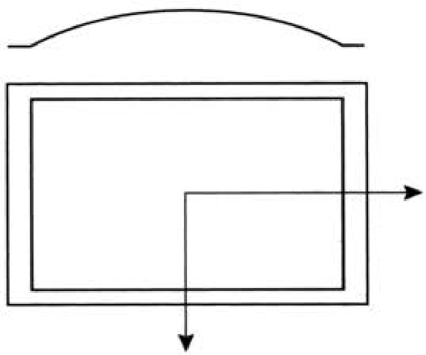


Figure B.2 Catenary shell of double curvature. (Shell formed by stretching a piece of hessian [flexible fabric] across a mould.)

B.2 USES OF FUNICULAR SHELLS

Funicular shells can be used in two ways. First, they can be used to cover large areas as a single shell (Figure B.1). Secondly, small funicular shells of about 1 m × 1 m can be precast and used for waffle slabs, also known as ribbed or voided slabs described in Clause 30 of IS 456 [1] (Figure B.2). The shape of these small shells can be got either mathematically or by actually stretching a membrane and loading it [2],[3]. Moulds are made of this shape and the small shells can be precast in large numbers to be used for these waffle slabs.

Reference [2] gives the equations for funicular shells over different plans. The equation to the surface of a funicular shell over an ellipse of dimensions a and b is as follows [2]:

$$z = \frac{w}{2\bar{N}}\left(\frac{a^2b^2}{a^2+b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

where, w = Loading on shell per unit area

$\bar{N}$ = Desired compressive stress on projected unit area

a and b Semi-major and semi-minor axes

For a circle, $a = b$ = radius of the circle.

Thus, with this shape we can build a funicular dome with no tension. Reference [2] gives also the illustration of a funicular shell roof over an oval ground plan for an assembly hall.

B.3 ANALYSIS OF LARGE FUNICULAR SHELLS

Funicular shell roofs can be made for any shape in plan. The methods of analysis are:

1. Elementary arch analysis in X- and Y-directions
2. By theory of shells
3. By relaxation method

Reference [2] gives the details of analysis of these shells.

Appendix C

GEOMETRIC CURVES

C.1 INTRODUCTION

Many of the concrete shells are curved surfaces obtained from translation and rotation of geometric curves. This helps the formulation of the mathematical theory of shells. It also helps us to construct the formwork of the shell accurately. In this appendix, we examine the common types of geometrical curves from which most of the concrete shells are derived. (Shells of arbitrary shapes have to be analysed by model analysis or modern finite element method.)

C.2 CONICS

Conics are curves obtained by cutting a right circular cone by a plane not passing through its vertex as shown in Figure C.1. A horizontal plane parallel to base gives a circle. An inclined plane not passing through the vertex gives an ellipse. A very much inclined plane cutting through the continuation of the cone also gives a hyperbola.

The commonly used shell surfaces are the members of the conics. They are formed by the translation or rotation of these curves. We examine the equations for these curves and some of the surfaces formed by these curves in the following sections.

C.3 EQUATIONS TO CONICS

We define ellipse, parabola hyperbola and circle with reference to the value of a constant ratio *e,* which varies with the conic. It is the ratio of the length of a point on the curve from the focus of the curve to the length of the point from the directrix as in Fig. C.3(b). The following are the more common curves.

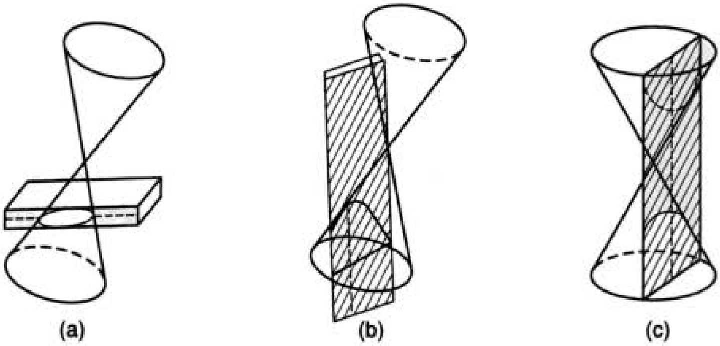


Figure C.1 Conics—Curves formed by cutting a cone with a plane: (a) Circle and ellipse, (b) Parabola, and (c) Hyperbola.

C.3.1 Ellipse ($e < 1$) (Figure C.2)

The value of SP/SM = e is less than 1 ($e < 1$).

The focus is at ae, 0 from origin.

Semi-major axis = a, semi-minor axis = b.

$$b^2 = a^2(1 - e^2)$$

Equation to the ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It is good to remember that an ellipse can be easily constructed by fixing the ends of a string of length $2a$ at ae, 0 and $-ae$, 0 and rotating a pencil placed in the string.

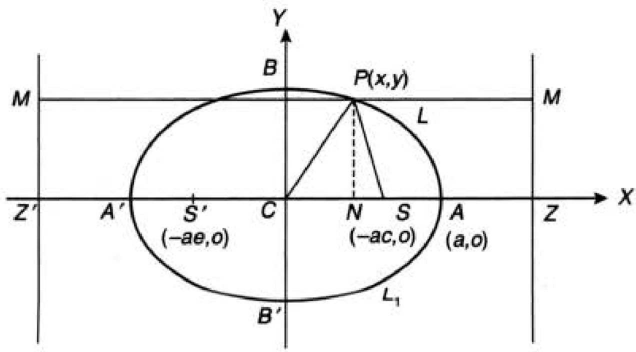


Figure C.2 Ellipse ($e < 1$).

C.3.2 Parabola ($e = 1$) (Figure C.3)

$$SP/PM = e; \text{ as } e = 1, SP = PM.$$

The equation for parabola is $y^2 = 4a(x - a)$.

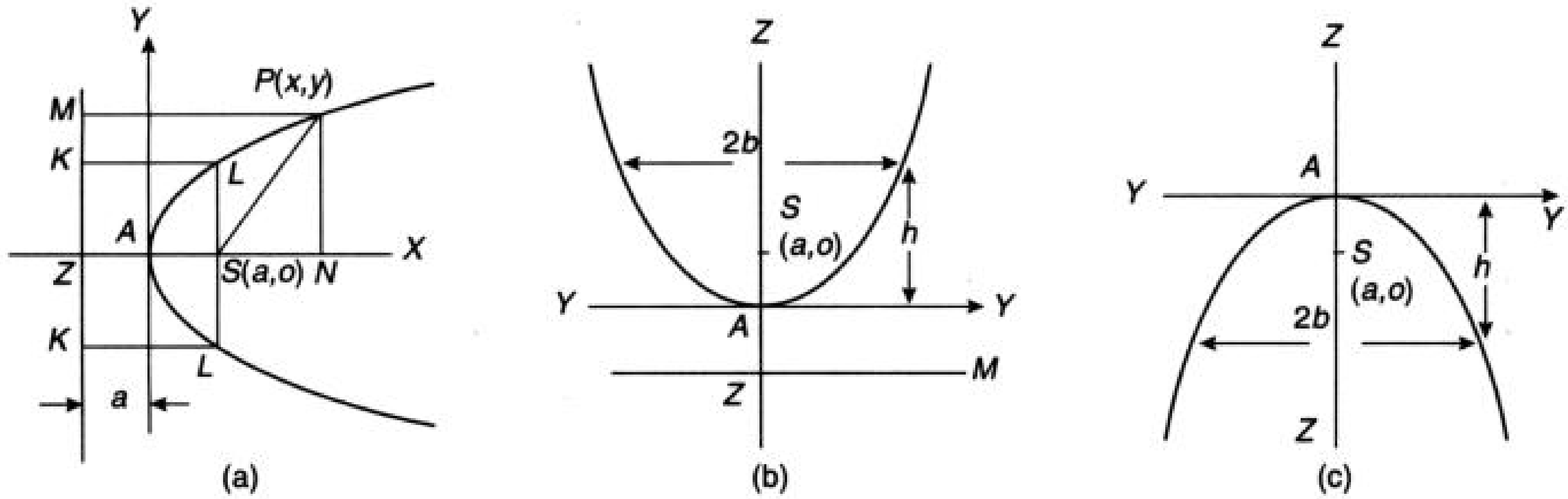


Figure C.3 Parabola (e = 1): (a) Conventional type, (b) Parabola with upward curvature, and (c) Parabola with downward curvature with Z- and Y-axes.

By changing the origin to the vertex of the parabola $(x - a)$, we get its equation as,

$$y^2 = 4ax$$

For a point $x = h$ and $y = b$, the equation becomes

$$b^2 = 4ah \qquad \text{or} \qquad 4a = \left(\frac{b^2}{h}\right)$$

$$\text{Equation to parabola} = y^2 = \frac{b^2 x}{h}$$

i.e.

$$x = h\left(\frac{y^2}{b^2}\right)$$

We can have parabolas with downward curvature and upward curvature as shown in Figure C.4.

(a) *Parabola with upward curvature:* With vertical axis as z and height at midpoint h and total length $2b$ along the horizontal, equation to parabola with upward curvature will be,

$$\text{With axis at top, } z = h\left(\frac{y^2}{b^2}\right)$$

(b) *Parabola with upward curvature:* The equation for this parabola with origin at the top point, as shown in Figure C.4, will be,

$$z = -h\left(\frac{y^2}{b^2}\right)$$

C.3.3 Hyperbola (e > 1) (Figure C.4)

Equation with reference to the XY-axis,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Hyperboloid shells are rotational shells formed by hyperbola. Paraboloids are translational shells formed by parabolas of same curvature.

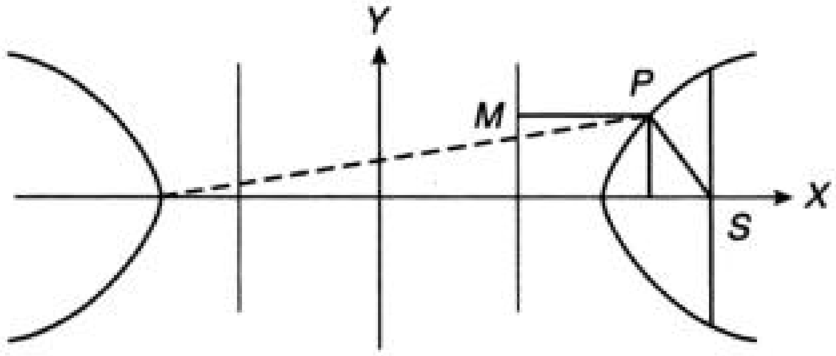


Figure C.4 Hyperbolas ($e > 1$).

C.3.4 Circle ($e = 0$) and Cylindrical Shells

The equation to a circle is well known as,

$$x^2 + y^2 = a^2$$

A cylindrical shell is formed by the translation of a straight line over the same circle at its two ends.

C.4 PARABOLOIDS

Paraboloids are formed by translation of one parabola over another parabola. Two unequal parabolas of the same sign of curvature will produce an elliptical paraboloid. Two equal parabolas of the same sign of curvature will produce a circular paraboloid. (Refer Chapter 11.)

(a) *Elliptic and circular paraboloids:* These are translational shells formed by two parabolas, one with z- and x-axes and the other with z- and y-axes, both with the same curvature (usually pointing downwards). Taking origin at top of the shell, the equation to the two parabolas will be,

$$z = h_x\left(\frac{x^2}{l_x^2}\right) \text{ and } z = h_y\left(\frac{y^2}{l_y^2}\right)$$

Hence, the equation to the surface of the shell will be,

$$z = h_x\left(\frac{x^2}{L_x^2}\right) + h_y\left(\frac{y^2}{L_y^2}\right)$$

When $h_1 = h_2$ and $L_x = L_y$, we get a circular hyperboloid surface.

(b) *Hyperbolic paraboloid shells:* When the parabolas are of opposing curvatures, we get a hyperbolic paraboloid as show in Figure 11.2.

The equation to the parabola will be,

$$z_1 = h_x\left(\frac{x^2}{L_x^2}\right) \text{ and } z_2 = -h_y\left(\frac{y^2}{L_y^2}\right)$$

The equation to hyperbolic paraboloids with curved edges will be,

$$z = h_x\left(\frac{x^2}{L_x^2}\right) - h_y\left(\frac{y^2}{L_y^2}\right)$$

[The equation to the hyperbolic paraboloid with straight edges as derived by Eq. (11.1) of Chapter 11 is, $z = \frac{h}{ab}xy$]

(c) *Groined parabolic vaults:* In Chapter 13, we have groined vaults formed by two parabolas of opposite curvatures.

The equation for the surface with origin at top of the shell pointing down will be as given by Eq. (13.1), and shown in Figure 13.3.

$$z = h_y\left(\frac{y^2}{L_y^2}\right) - h_x\left(\frac{x^2}{L_x^2}\right)$$

C.5 HYPERBOLOID OF REVOLUTION

If a hyperbola *rotates* about the vertical axis (z-axis), we get a hyperboloid as shown in Figure C.5.

The general equation to the surface as given in Reference [1] is,

$$\frac{x^2}{e^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$$

The equation to the popularly used cooling tower surface where $e = a$ becomes,

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{b^2} = 1$$

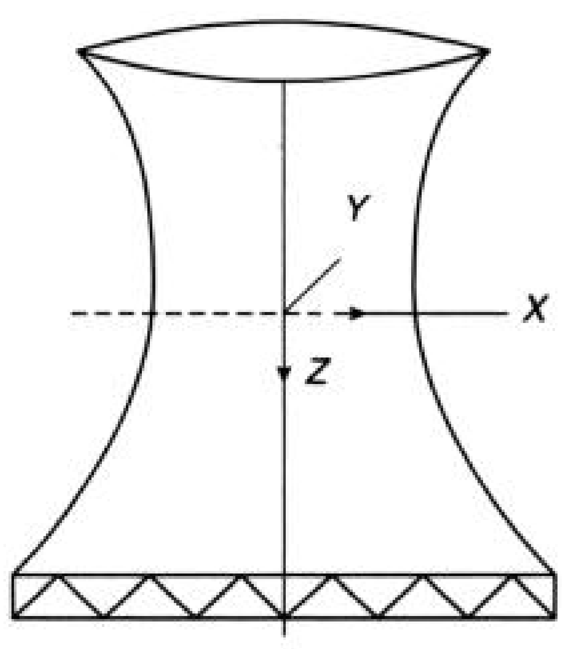


Figure C.5 Hyperboloid of revolution of one sheet (cooling tower).

References [1] and [2] give details of analysis of such structures.

SUMMARY

A short discussion on the shape of the different geometrical shapes used for the formation of the popular shell roofs is given in this Appendix.

REFERENCES

[1] Chatterjee, N.K., *Theory and Design of Concrete Shells*, Oxford and IBM, Calcutta, 1971.
[2] Ramaswamy, G.S., *Design and Construction of Shell Roofs*, McGraw-Hill, New York, 1968.

Appendix D

TENSION STRUCTURES

D.1 INTRODUCTION

Concrete is strong in compression and weak in tension. Thus, the shapes we choose for concrete shell roofs are those in which most of the surfaces are in compression when used as a roof.

However, there are another set of structures such as the circus tents, camp tents made of stretched waterproof cloth stretched over ropes and tensioned from their edges. These structures where the roof materials are in tension are called *tension structures*.

Recently, much work has gone into the development of special waterproof materials (fabrics) which can take large tension stresses. These materials are used for tension structures. They can be provided with eyelets at their ends to tie them to steel frames. Such tension structures are very popular for temporary structures such as those for conferences or car sheds attached to a residential building.

D.2 TENSION STRUCTURES

There are broadly two types of tension structures as described below.

D.2.1 Tension Membrane Structures

Temporary assembly halls for religious, political or social gatherings, structures for exhibitions, car parking or even temporary or semi-permanent small car sheds can be made of these materials with eyelets provided at the edges and stretched over tubular steel frames as shown in Figures D.1 and D.2. The material is in tension and tied on to the steel frames at the ends and supported on the steel frames along the interiors along specified lines. Figure D.2 shows a fabricated car temporary shed near a residence.

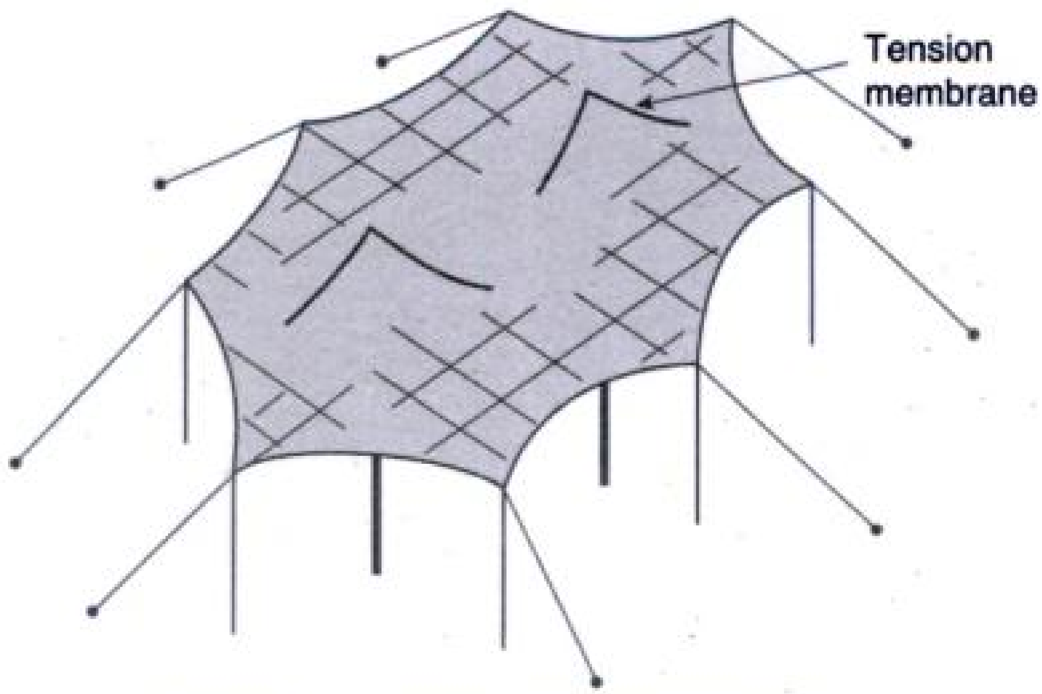


Figure D.1 Tension membrane structure.

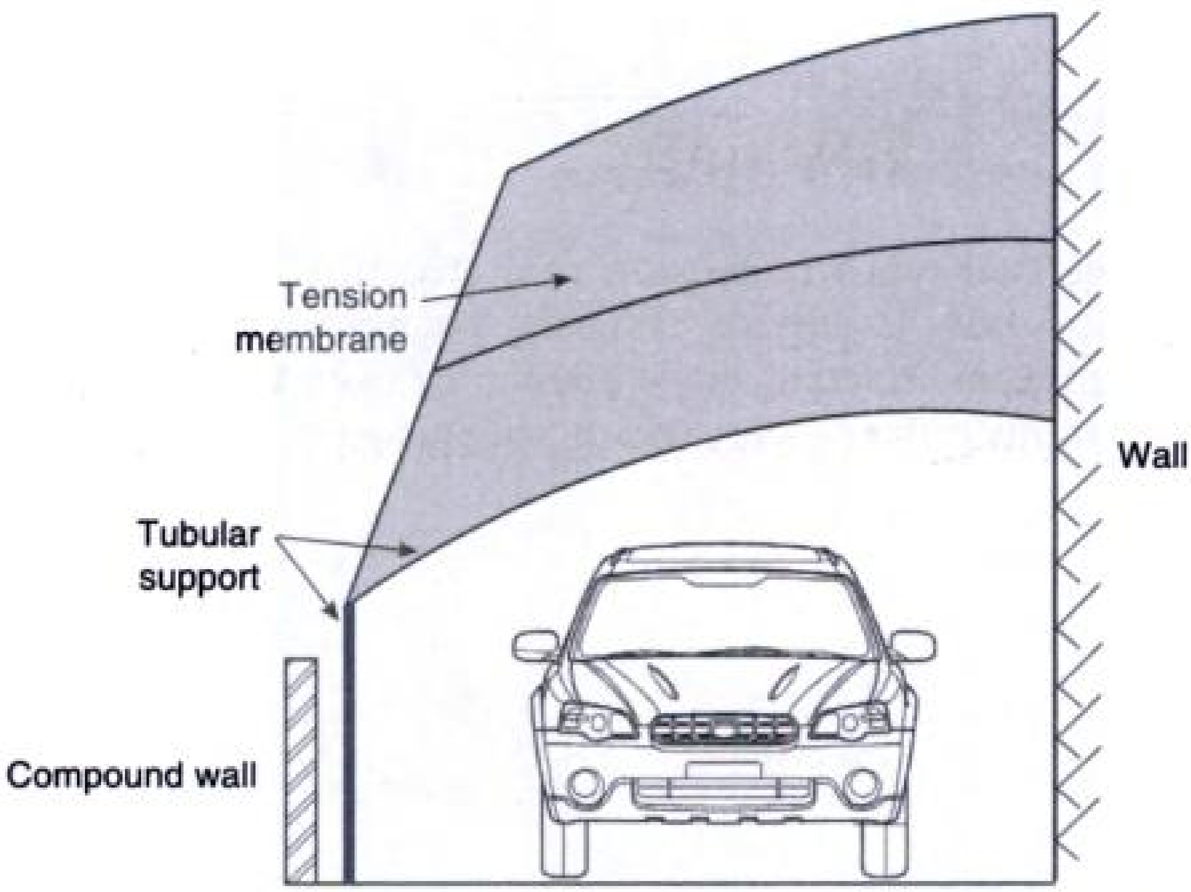


Figure D.2 Tension membrane used for a car shed.

These tension structures can be made of all shapes and made attractive. Data on the different shapes and types that can be used for different purposes can be obtained from the internet. Different types of fabrics suitable for these stretched tension structures are available in the market.

D.2.2 Tension Cable Structures

Tension cable structures consist of arch and cable system that support the roof covering as shown in Figure D.3.

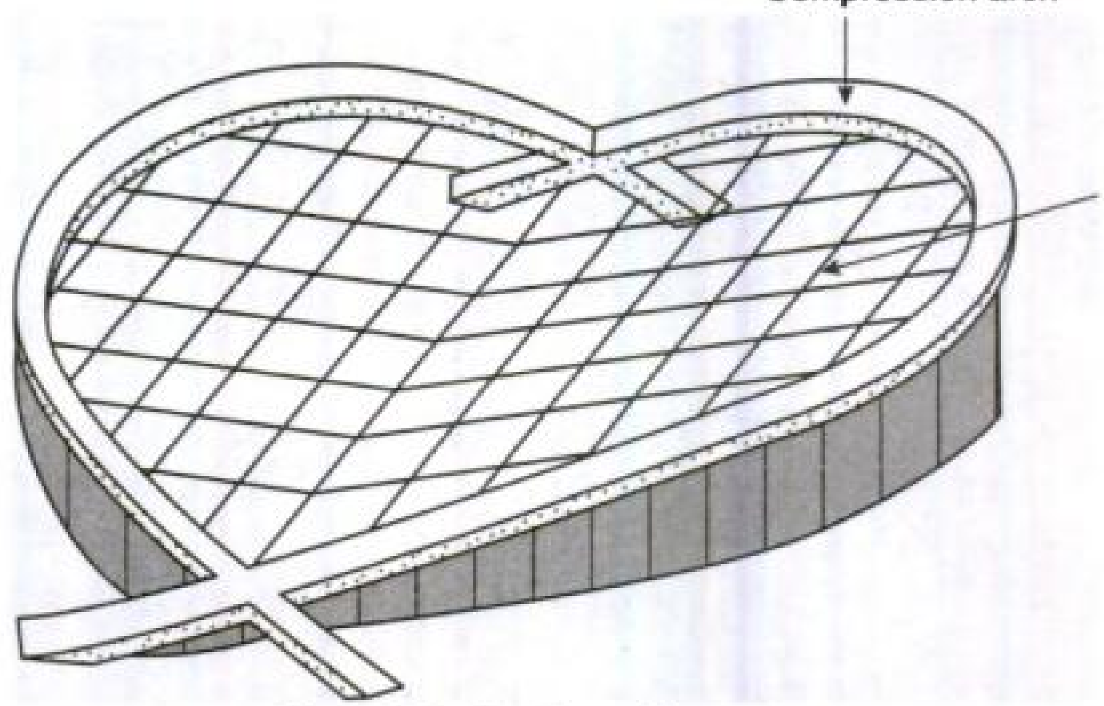


Figure D.3 Tension cable structure.

SUMMARY

Only a brief mentions of the two types of tension roof structures is presented in this appendix. The aim of this appendix is only to bring the attention of the reader to such structures which are different from concrete shells but can be used for the same purpose of "covering large space without pillars" with which we started our discussions in this book.

Appendix E

AVAILABLE TABLES FOR DESIGN OF REINFORCED CONCRETE SHELLS

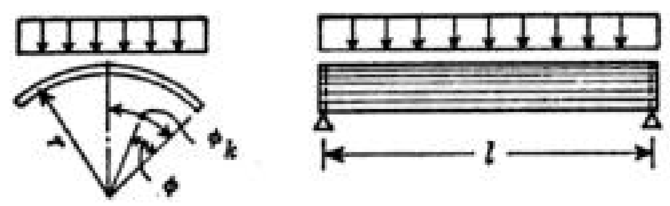
E.1 ASCE MANUAL NO. 31 TABLES FOR DESIGN OF CYLINDRICAL CONCRETE SHELL ROOFS*

* Both of Tables published with permission from American Society of Civil Engineers.

TABLES FROM ASCE MANUAL NO: 31
(DESIGN OF CYLINDRICAL CONCRETE SHELL ROOFS)

TABLE 1A.—MEMBRANE FORCES AND DISPLACEMENTS IN SIMPLY SUPPORTED CYLINDRICAL SHELLS; LOADS UNIFORMLY DISTRIBUTED ALONG THE LENGTH OF THE BARREL

(a) UNIFORM TRANSVERSE LOAD[a]



Longitudinal Force T_x—

$$p_u r \left(\frac{l}{r}\right)^2 \times \frac{x}{l}\left(1 - \frac{x}{l}\right) \times \text{Col. (1)}$$

Shearing Force S—

$$p_u r \left(\frac{l}{r}\right) \times \left(1 - \frac{2x}{l}\right) \times \text{Col. (2)}$$

Transverse Force T_ϕ—

$$p_u r \times \text{Col. (3)}$$

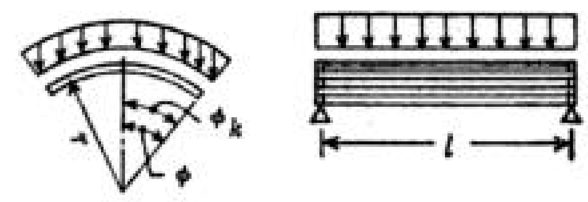
Vertical Displacement[b] Δ_V—

$$p_u r \times \frac{l^4}{r^3 t E} \times \left\{2\left(\frac{r}{l}\right)^4 + \left(\frac{x}{l}\right)^4 - 2\left(\frac{x}{l}\right)^3 + \frac{x}{l} + 6\left(\frac{r}{l}\right)^2\left[\frac{x}{l} - \left(\frac{x}{l}\right)^2\right]\right\} \times \text{Col. (4)}$$

Horizontal Displacement[c] Δ_H—

$$p_u r \frac{l^4}{r^3 t E}\left\{\left(\frac{r}{l}\right)^4 \times \text{Col. (5)} + \left[2\left(\frac{r}{l}\right)^4 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4 + \frac{x}{l} + 6\left(\frac{r}{l}\right)^2\left(\frac{x}{l} - (x/l)^2\right)\right] \times \text{Col. (6)}\right\}$$

(b) DEAD WEIGHT LOAD[a]



Longitudinal Force T_x—

$$p_d r \left(\frac{l}{r}\right)^2 \times \frac{x}{l}\left(1 - \frac{x}{l}\right) \times \text{Col. (7)}$$

Shearing Force S—

$$p_d r \left(\frac{l}{r}\right) \times \left(1 - \frac{2x}{l}\right) \times \text{Col. (8)}$$

Transverse Force T_ϕ—

$$p_d r \times \text{Col. (9)}$$

Vertical Displacement Δ_V—

$$p_d r \frac{l^4}{r^3 t E}\left\{-\frac{1}{6}\left(\frac{x}{l}\right)^3 + \frac{1}{12}\left(\frac{x}{l}\right)^4 + \frac{1}{12}\left(\frac{x}{l}\right) + 2\left(\frac{r}{l}\right)^2\left[\frac{x}{l} - \left(\frac{x}{l}\right)^2\right] + \left(\frac{r}{l}\right)^4 \times \text{Col. (10)}\right\}$$

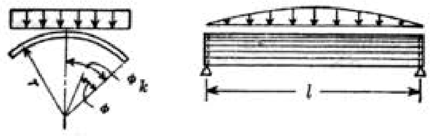
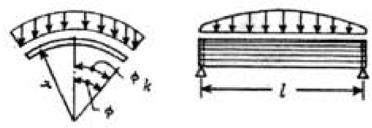
Horizontal Displacement Δ_H—

$$p_d r \frac{l^4}{r^3 t E}\left(\frac{r}{l}\right)^4 \times \text{Col. (11)}$$

$\phi_k - \phi$	T_x	S	T_ϕ	Δ_V	Δ_H		T_x	S	T_ϕ	Δ_V	Δ_H
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
0	−1.500	0	−1.0000	0.5000	0	0	−1.0000	0	−1.0000	1.0000	0
5	−1.477	−0.1302	−0.9924	0.4944	0.0872	−0.0003	−0.9962	−0.0871	−0.9962	0.9924	0.0868
10	−1.409	−0.2565	−0.9698	0.4775	0.1736	−0.0026	−0.9848	−0.1736	−0.9848	0.9698	0.1710
15	−1.295	−0.3749	−0.9330	0.4506	0.2588	−0.0072	−0.9659	−0.2589	−0.9659	0.9330	0.2500
20	−1.150	−0.4820	−0.8830	0.4148	0.3420	−0.0200	−0.9397	−0.3421	−0.9397	0.8830	0.3214
25	−0.965	−0.5746	−0.8214	0.3722	0.4226	−0.0377	−0.9063	−0.4225	−0.9063	0.8214	0.3830
30	−0.750	−0.6503	−0.7500	0.3447	0.5000	−0.0625	−0.8660	−0.5000	−0.8660	0.7500	0.4330
35	−0.513	−0.7048	−0.6710	0.2748	0.5736	−0.0944	−0.8191	−0.5737	−0.8191	0.6710	0.4698
40	−0.261	−0.7385	−0.5868	0.2247	0.6428	−0.1328	−0.7660	−0.6428	−0.7660	0.5868	0.4924
45	0	−0.7500	−0.5000	0.1767	0.7071	−0.1768	−0.7071	−0.7071	−0.7071	0.5000	0.5000
50	0.261	−0.7385	−0.4132	0.1328	0.7660	−0.2247	−0.6428	−0.7661	−0.6428	0.4132	0.4924
55	0.513	−0.7048	−0.3290	0.0944	0.8191	−0.2749	−0.5736	−0.8192	−0.5736	0.3290	0.4698
60	0.750	−0.6503	−0.2500	0.0625	0.8660	−0.3248	−0.5000	−0.8660	−0.5000	0.2500	0.4330
65	0.965	−0.5746	−0.1786	0.0377	0.9063	−0.3722	−0.4226	−0.9062	−0.4226	0.1786	0.3830
70	1.150	−0.4820	−0.1170	0.0199	0.9397	−0.4149	−0.3420	−0.9397	−0.3420	0.1170	0.3214
75	1.295	−0.3749	−0.0669	0.0086	0.9659	−0.4506	−0.2588	−0.9659	−0.2588	0.0669	0.2500
80	1.409	−0.2565	−0.0301	0.0026	0.9848	−0.4776	−0.1736	−0.9847	−0.1736	0.0301	0.1710
85	1.477	−0.1302	−0.0076	0.0004	0.9962	−0.4943	−0.0872	−0.9962	−0.0872	0.0076	0.0868
90	1.500	0	0	0	1.0000	−0.5000	0	−1.0000	0	0	0

[a] The use of column numbers in the formulas refers to the appropriate coefficient in the column cited
[b] Downward direction is positive. [c] Inward direction is positive.

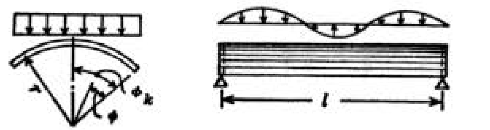
TABLE 1B.—Membrane Forces and Displacements in Simply Supported Cylindrical Shells; Loads Varying Longitudinally from Zero at the Ends to Maximum Positive at the Middle ($n = 1$)

(a) Uniform Transverse Load	(b) Dead Weight Load
	
Longitudinal Force T_x— $p_u r \left[\left(\frac{l}{r}\right)^2 \times \text{Col. (1)}\right] \sin\frac{\pi x}{l}$	Longitudinal Force T_x— $p_d r \left[\left(\frac{l}{r}\right)^2 \times \text{Col. (7)}\right] \sin\frac{\pi x}{l}$
Shearing Force S— $p_u r \left[\left(\frac{l}{r}\right) \times \text{Col. (2)}\right] \cos\frac{\pi x}{l}$	Shearing Force S— $p_d r \left[\left(\frac{l}{r}\right) \times \text{Col. (8)}\right] \cos\frac{\pi x}{l}$
Transverse Force T_ϕ— $p_u r \times \text{Col. (3)} \times \sin\frac{\pi x}{l}$	Transverse Force T_ϕ— $p_d r \times \text{Col. (9)} \times \sin\frac{\pi x}{l}$
Vertical Displacement Δ_V— $p_u r \frac{l^4}{r^3 t E}\left[\left(1+\frac{1}{2}(\pi r/l)^2+\frac{1}{12}(\pi r/l)^4\right) \times \text{Col. (4)}\right] \sin\frac{\pi x}{l}$	Vertical Displacement Δ_V— $p_d r \frac{l^4}{r^3 t E}\left[\left(\frac{2r}{\pi l}\right)^2+\frac{2}{\pi^4} + \left(\frac{r}{l}\right)^4 \times \text{Col. (10)}\right] \sin\frac{\pi x}{l}$
Horizontal Displacement Δ_H— $+p_u r \frac{l^4}{r^3 t E}\left\{\left(\frac{r}{l}\right)^4 \times \text{Col. (5)} + \left[1+\frac{1}{2}\left(\frac{\pi r}{l}\right)^2 + \frac{1}{12}\left(\frac{\pi r}{l}\right)^4\right] \times \text{Col. (6)}\right\} \times \sin\frac{\pi x}{l}$	Horizontal Displacement Δ_H— $+p_d r \frac{l^4}{r^3 t E}\left[\left(\frac{r}{l}\right)^4 \times \text{Col. (11)}\right] \times \sin\frac{\pi x}{l}$

$\phi_k - \phi$	T_x	S	T_ϕ	Δ_V	Δ_H		T_x	S	T_ϕ	Δ_V	Δ_H
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
0	−0.3040	0	−1.0000	0.12319	0	0	−0.2026	0	−1.000	1.0000	0
5	−0.2993	−0.0829	−0.9924	0.12180	0.0872	−0.00009	−0.2019	−0.0555	−0.9962	0.9924	0.0868
10	−0.2856	−0.1633	−0.9698	0.11766	0.1736	−0.00064	−0.1996	−0.1105	−0.9848	0.9698	0.1710
15	−0.2623	−0.2387	−0.9330	0.11102	0.2588	−0.00213	−0.1957	−0.1648	−0.9659	0.9330	0.2500
20	−0.2329	−0.3069	−0.8830	0.10222	0.3420	−0.00490	−0.1904	−0.2178	−0.9397	0.8830	0.3214
25	−0.1954	−0.3658	−0.8214	0.09170	0.4226	−0.00930	−0.1837	−0.2690	−0.9063	0.8214	0.3830
30	−0.1520	−0.4140	−0.7500	0.08001	0.5000	−0.01539	−0.1754	−0.3183	−0.8660	0.7500	0.4330
35	−0.1040	−0.4487	−0.6710	0.06771	0.5736	−0.02325	−0.1660	−0.3652	−0.8191	0.6710	0.4698
40	−0.0528	−0.4702	−0.5868	0.05537	0.6428	−0.03272	−0.1552	−0.4092	−0.7660	0.5868	0.4924
45	0	−0.4775	−0.5000	0.04355	0.7071	−0.04355	−0.1433	−0.4502	−0.7071	0.5000	0.5000
50	0.0528	−0.4702	−0.4132	0.03272	0.7660	−0.05537	−0.1302	−0.4877	−0.6428	0.4132	0.4924
55	0.1040	−0.4487	−0.3290	0.02325	0.8191	−0.06771	−0.1162	−0.5215	−0.5736	0.3290	0.4698
60	0.1520	−0.4140	−0.2500	0.01539	0.8660	−0.08001	−0.1013	−0.5513	−0.5000	0.2500	0.4330
65	0.1954	−0.3658	−0.1786	0.00930	0.9063	−0.09170	−0.0856	−0.5769	−0.4226	0.1786	0.3830
70	0.2329	−0.3069	−0.1170	0.00490	0.9397	−0.10222	−0.0693	−0.5982	−0.3420	0.1170	0.3214
75	0.2623	−0.2387	−0.0669	0.00213	0.9659	−0.11102	−0.0524	−0.6149	−0.2588	0.0669	0.2500
80	0.2856	−0.1633	−0.0301	0.00064	0.9848	−0.11766	−0.0351	−0.6269	−0.1736	0.0301	0.1710
85	0.2993	−0.0829	−0.0076	0.00009	0.9962	−0.12180	−0.0177	−0.6342	−0.0872	0.0076	0.0868
90	0.3040	0	0	0	1.0000	−0.12319	0	−0.6366	0	0	0

TABLE 1C.—Membrane Forces and Displacements in Simply Supported Cylindrical Shells; Loads Varying Sinusoidally from Zero at the Ends to Maximum at the Middle ($n = 3$)

(a) Uniform Transverse Load



Longitudinal Force T_x—

$$p_u r \left(\frac{l}{r}\right)^2 \times \text{Col. (1)} \times \sin\frac{3\pi x}{l}$$

Shearing Force S—

$$p_u r \frac{l}{r} \times \text{Col. (2)} \times \cos\frac{3\pi x}{l}$$

Transverse Force T_ϕ—

$$p_u r \times \text{Col. (3)} \times \sin\frac{3\pi x}{l}$$

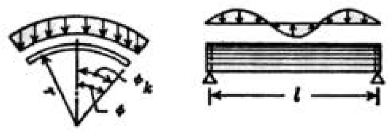
Vertical Displacement Δ_V—

$$p_u r \frac{l^4}{r^3 t E}\left[1 + \frac{9}{2}\left(\frac{\pi r}{l}\right)^2 + \frac{81}{12}\left(\frac{\pi r}{l}\right)^4\right] \text{Col. (4)} \times \sin\frac{3\pi x}{l}$$

Horizontal Displacement Δ_H—

$$p_u r \frac{l^4}{r^3 t E}\left\{\left(\frac{r}{l}\right)^4 \times \text{Col. (5)} + \left[1 + \frac{9}{2}\left(\frac{\pi r}{l}\right)^4 + \frac{81}{12}\left(\frac{\pi r}{l}\right)^4\right] \text{Col. (6)}\right\} \sin\frac{3\pi x}{l}$$

(b) Dead Weight Load



Longitudinal Force T_x—

$$p_d r \left(\frac{l}{r}\right)^2 \times \text{Col. (7)} \times \sin\frac{3\pi x}{l}$$

Shearing Force S—

$$p_d r \left(\frac{l}{r}\right) \times \text{Col. (8)} \times \cos\frac{3\pi x}{l}$$

Transverse Force T_ϕ—

$$p_d r \times \text{Col. (9)} \times \sin\frac{3\pi x}{l}$$

Vertical Displacement Δ_V—

$$p_d r \frac{l^4}{r^3 t E}\left[\frac{2}{81(\pi)^4} + \frac{4}{9}\left(\frac{r}{\pi l}\right)^2 + \left(\frac{r}{l}\right)^4 \times \text{Col. (10)}\right] \sin\frac{3\pi x}{l}$$

Horizontal Displacement Δ_H—

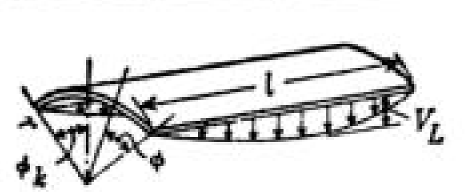
$$p_d r \frac{l^4}{r^3 t E}\left(\frac{r}{l}\right)^4 \times \text{Col. (11)} \times \sin\frac{3\pi x}{l}$$

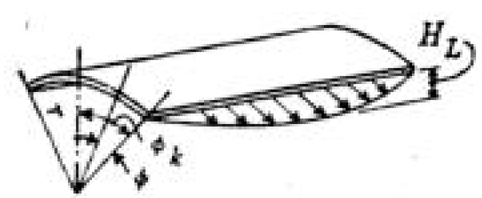
$\phi_k - \phi$	T_x	S	T_ϕ	Δ_V	Δ_H		T_x	S	T_ϕ	Δ_V	Δ_H
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
0	−0.0338	0	−1.0000	0.00152	0	0	−0.0225	0	−1.0000	1.0000	0
5	−0.0332	−0.0276	−0.9924	0.00150	0.0872	−0.00000	−0.0224	−0.0185	−0.9962	0.9924	0.0868
10	−0.0317	−0.0544	−0.9698	00.0145	0.1736	−0.00000	−0.0222	−0.0368	−0.9848	0.9698	0.1710
15	−0.0291	−0.0796	−0.9330	0.00137	0.2588	−0.00002	−0.0219	−0.0549	−0.9659	0.9330	0.2500
20	−0.0259	−0.1023	−0.8830	0.00126	0.3420	−0.00006	−0.0212	−0.0726	−0.9397	0.8830	0.3214
25	−0.0217	−0.1219	−0.8214	0.00113	0.4226	−0.00011	−0.0204	−0.0897	−0.9063	0.8214	0.3830
30	−0.0169	−0.1380	−0.7500	0.00099	0.5000	−0.00019	−0.0195	−0.1061	−0.8660	0.7500	0.4330
35	−0.0116	−0.1496	−0.6710	0.00084	0.5736	−0.00029	−0.0184	−0.1217	−0.8191	0.6710	0.4698
40	−0.0059	−0.1567	−0.5868	0.00068	0.6428	−0.00040	−0.0172	−0.1364	−0.7660	0.5868	0.4924
45	0	−0.1592	−0.5000	0.00054	0.7071	−0.00054	−0.0159	−0.1501	−0.7071	0.5000	0.5000
50	0.0059	−0.1567	−0.4132	0.00040	0.7660	−0.00068	−0.0145	−0.1626	−0.6428	0.4132	0.4924
55	0.0116	−0.1496	−0.3290	0.00029	0.8191	−0.00084	−0.0129	−0.1738	−0.5736	0.3290	0.4698
60	0.0169	−0.1380	−0.2500	0.00019	0.8660	−0.00099	−0.0113	−0.1838	−0.5000	0.2500	0.4330
65	0.0217	−0.1219	−0.1786	0.00011	0.9063	−0.00113	−0.0095	−0.1923	−0.4226	0.1786	0.3830
70	0.0259	−0.1023	−0.1170	0.00006	0.9397	−0.00126	−0.0077	−0.1994	−0.3420	0.1170	0.3214
75	0.0291	−0.0796	−0.0669	0.00002	0.9659	−0.00137	−0.0058	−0.2050	−0.2588	0.0669	0.2500
80	0.0317	−0.0544	−0.0301	0.00000	0.9848	−0.00145	−0.0040	−0.2090	−0.1736	0.0301	0.1710
85	0.0332	−0.0276	−0.0076	0.00000	0.9962	−0.00150	−0.0019	−0.2114	−0.0872	0.0076	0.0868
90	0.0338	0	0	0	1.0000	−0.00152	0	−0.2122	0	0	0

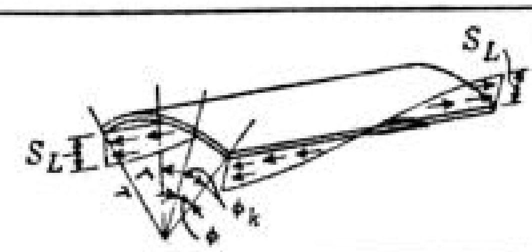
TABLE 2A.—Symmetrical Edge Loads on Simply Supported Cylindrical Shells ($n = 1$)

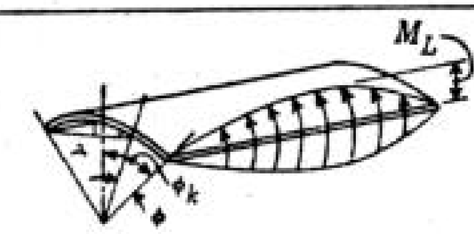
(a) Basic Formulas and Loading Diagrams

Vertical Edge Load	Horizontal Edge Load	Shear Edge Load	Edge Moment Load
Longitudinal Force T_x— $V_L\left[\left(\frac{l}{r}\right)^2 \times \text{Col. (1)}\right]\sin\frac{\pi x}{l}$	Longitudinal Force T_x— $H_L\left[\left(\frac{l}{r}\right)^2 \times \text{Col. (5)}\right]\sin\frac{\pi x}{l}$	Longitudinal Force T_x— $S_L\left[\left(\frac{l}{r}\right)^2 \times \text{Col. (9)}\right]\sin\frac{\pi x}{l}$	Longitudinal Force T_x— $\frac{M_L}{r}\left[\left(\frac{l}{r}\right)^2 \times \text{Col. (13)}\right]\sin\frac{\pi x}{l}$
Shearing Force S— $V_L\left[\frac{l}{r} \times \text{Col. (2)}\right]\cos\frac{\pi x}{l}$	Shearing Force S— $H_L\left[\frac{l}{r} \times \text{Col. (6)}\right]\cos\frac{\pi x}{l}$	Shearing Force S— $S_L\left[\frac{l}{r} \times \text{Col. (10)}\right]\cos\frac{\pi x}{l}$	Shearing Force S— $\frac{M_L}{r}\left[\frac{l}{r} \times \text{Col. (14)}\right]\cos\frac{\pi x}{l}$
Transverse Force T_ϕ— $V_L \times \text{Col. (3)} \times \sin\frac{\pi x}{l}$	Transverse Force T_ϕ— $H_L \times \text{Col. (7)} \times \sin\frac{\pi x}{l}$	Transverse Force T_ϕ— $S_L \times \text{Col. (11)} \times \sin\frac{\pi x}{l}$	Transverse Force T_ϕ— $\frac{M_L}{r} \times \text{Col. (15)} \times \sin\frac{\pi x}{l}$
Transverse Moment M_ϕ— $V_L\,[r \times \text{Col. (4)}]\sin\frac{\pi x}{l}$	Transverse Moment M_ϕ— $H_L\,[r \times \text{Col. (8)}]\sin\frac{\pi x}{l}$	Transverse Moment M_ϕ— $S_L\,[r \times \text{Col. (12)}]\sin\frac{\pi x}{l}$	Transverse Moment M_ϕ— $M_L \times \text{Col. (16)} \times \sin\frac{\pi x}{l}$









(b) $r/t = 100$ and $r/l = 0.1$

	Vertical Edge Load				Horizontal Edge Load				Shear Edge Load				Edge Moment Load			
ϕ	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
$\phi_k = 30$:																
30	− 5.382	0	−3.471	−0.3278	− 0.0013	0	+0.9997	+0.1310	−0.0904	0	−0.0207	−0.0010	− 0.0038	0	+ 0.0008	+ 0.9738
20	− 3.575	−2.619	−2.656	−0.2779	+ 0.0017	−0.0001	+0.9848	+0.1162	−0.0396	−0.0403	−0.0081	−0.0007	+ 0.0172	+ 0.0021	+ 0.0022	+ 0.9767
10	+ 1.809	−3.265	−0.775	−0.1512	+ 0.0034	+0.0017	+0.9399	+0.0722	+0.1118	−0.0250	+0.0142	+0.0002	+ 0.0264	+ 0.0173	+ 0.0014	+ 0.9854
0	+10.69	0	+0.500	0	− 0.0151	0	+0.8660	0	+0.3600	+0.1000	0	0	− 0.1370	0	0	+ 1.000

TABLE 2A.—(Continued)

ϕ	VERTICAL EDGE LOAD				HORIZONTAL EDGE LOAD				SHEAR EDGE LOAD				EDGE MOMENT LOAD			
	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
(b) (Continued)																
$\phi_k = 35$:																
35	− 3.430	0	−2.959	−0.3703	− 0.0061	0	+0.9986	+0.1754	−0.0774	0	−0.0243	−0.0015	− 0.0310	0	− 0.0051	+0.9642
30	− 3.220	−0.921	−2.820	−0.3594	− 0.0043	−0.0068	+0.9951	+0.1717	−0.0694	−0.0205	−0.0213	−0.0014	− 0.0211	− 0.0076	− 0.0037	+0.9650
20	− 1.538	−2.302	−1.812	−0.2781	+ 0.0060	−0.0136	+0.9663	+0.1425	−0.0054	−0.0439	−0.0015	−0.0008	+ 0.0373	− 0.0043	+ 0.0040	+0.9710
10	+ 1.822	−2.301	−0.330	−0.1444	+ 0.0080	+0.0041	+0.9071	+0.0848	+0.1212	−0.0149	+0.0185	−0.0002	+ 0.0469	+ 0.0251	+ 0.0045	+0.9825
0	+ 6.845	0	+0.574	0	− 0.0316	0	+0.8192	0	+0.3076	+0.1000	0	0	− 0.2051	0	0	+1.000
$\phi_k = 40$:	+															
40	− 2.210	0	−2.534	−0.4048	− 0.0165	0	+0.9954	+0.2247	−0.0675	0	−0.0279	−0.0022	− 0.0712	0	− 0.0184	+0.9530
30	− 1.806	−1.138	−2.168	−0.3683	− 0.0049	−0.0068	+0.9836	+0.2041	−0.0461	−0.0331	−0.0179	−0.0018	− 0.0196	− 0.0293	− 0.0074	+0.9570
20	− 0.578	−1.830	−1.205	−0.2691	+ 0.0176	−0.0032	+0.9420	+0.1672	+0.0181	−0.0423	+0.0054	−0.0009	+ 0.0815	− 0.0118	+ 0 0112	+0.9653
10	+ 1.516	−1.614	−0.045	−0.1352	+ 0.0156	+0.0081	+0.8685	+0.1047	+0.1236	−0.0057	+0.0221	−0.0002	+ 0.0724	+ 0.0998	+ 0.0106	+0.9798
0	+ 4.534	0	+0.643	0	− 0.0631	0	+0.7660	0	+0.2682	+0.1000	0	0	− 0.3099	0	0	+1.000
$\phi_k = 45$:																
45	− 1.602	0	−2.231	−0.4355	− 0.0359	0	+0.9873	+0.2779	−0.0596	0	−0.0315	−0.0031	− 0.1280	0	− 0.0437	0.9392
40	− 1.546	−0.434	−2.163	−0.4274	− 0.0309	−0.0094	+0.9849	+0.2743	−0.0559	−0.0160	−0.0292	−0.0030	− 0.1102	− 0.0325	− 0.0385	+0.9402
30	− 1.098	−1.180	−1.646	−0.3654	+ 0.0033	−0.0181	+0.9642	+0.2461	−0.0258	−0.0398	−0.0123	−0.0022	+ 0.0128	− 0.0642	− 0.0047	−0.9469
20	− 0.166	−1.550	−0.763	−0.2562	+ 0.0412	−0.0050	+0.9138	+0.1899	+0.0338	−0.0389	+0.0120	−0.0010	+ 0.1523	− 0.0158	+ 0.0277	+0.9595
10	+ 1.310	−1.263	+0.179	−0.1255	+ 0.0260	+0.1175	+0.8251	+0.1072	+0.1222	+0.0025	+0.0250	−0.0002	+ 0.0981	+ 0.0680	+ 0.0212	+0.9771
0	+ 3.407	0	+0.707	0	− 0.1185	0	+0.7071	0	+0.2378	+0.1000	0	0	− 0.4613	0	0	+1.000
$\phi_k = 50$:																
50	− 1.051	0	−1.931	−0.4541	− 0.0693	0	+0.9696	+0.3336	−0.0531	0	−0.0350	−0.0042	− 0.2048	0	− 0.0885	+0.9218
40	− 0.948	−0.558	−1.741	−0.4265	− 0.0213	−0.0326	+0.9650	+0.3198	−0.0423	−0.0271	−0.0268	−0.0037	− 0.1192	− 0.0964	− 0.0573	+0.9258
30	− 0.619	−0.999	−1.208	−0.3492	+ 0.0284	−0.0366	+0.9430	+0.2786	−0.0098	−0.0424	−0.0057	−0.0025	+ 0.0849	− 0.1086	+ 0.0111	+0.9371
20	− 0.001	−1.184	−0.451	−0.2379	+ 0.0820	−0.0039	+0.8851	+0.2104	+0.0440	−0.0340	+0.0181	−0.0010	+ 0.2503	− 0.0099	+ 0.0576	+0.9537
10	+ 0.998	−0.930	+0.317	−0.1141	+ 0.0374	+0.0357	+0.7738	+0.1167	+0.1188	+0.0077	+0.0275	−0.0002	+ 0.1176	+ 0.1119	+ 0.0373	+0.9745
0	+ 2.488	0	+0.766	0	− 0.2047	0	+0.6428	0	+0.2138	+0.1000	0	0	− 0.6695	0	0	+1.000
(c) $r/t = 100$ and $r/l = 0.2$																
$\phi_k = 30$:																
30	− 5.160	0	−3.420	−0.3028	− 0.0742	0	+0.9815	+0.1219	−0.1946	0	−0.0414	−0.0005	− 0.549	0	− 0.0813	+0.9009
20	− 3.440	−2.531	−2.632	−0.2565	− 0.0123	−0.0292	+0.9757	+0.1084	−0.0755	−0.0776	−0.0145	0	− 0.073	− 0.2142	− 0.0153	+0.9124
10	− 1.763	−3.228	−0.787	−0.1395	+ 0.0700	−0.0087	+0.9427	+0.0678	+0.2208	−0.0469	+0.0287	+0.0009	+ 0.557	− 0.0429	+ 0.0568	+0.9456
0	+10.92	0	+0.500	0	− 0.1140	0	+0.8660	0	+0.7119	+0.2000	0	0	− 1.067	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	Vertical Edge Load T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	Horizontal Edge Load T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	Shear Edge Load T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	Edge Moment Load T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
(c) (Continued)																
$\phi_k=35$:																
35	− 3.087	0	−2.871	−0.3347	− 0.1552	0	+0.9603	+0.1591	−0.1487	0	−0.0458	−0.0024	− 0.912	0	− 0.1914	+0.8637
30	− 2.939	−0.833	−2.744	−0.3248	− 0.1260	−0.0399	+0.9622	+0.1559	−0.1334	−0.0394	−0.0401	−0.0022	− 0.742	− 0.2343	− 0.1585	+0.8669
20	− 1.639	−2.156	−1.807	−0.2514	+ 0.0561	−0.0634	+0.9649	+0.1301	−0.0106	−0.0845	−0.0018	−0.0012	+ 0.327	− 0.3747	+ 0.0315	+0.8914
10	+ 1.501	−2.296	−0.365	−0.1302	+ 0.1526	+0.0074	+0.9224	+0.0781	+0.2378	−0.0280	+0.0369	−0.0001	+ 0.928	+ 0.0486	+ 0.1212	+0.9362
0	+ 7.434	0	+0.574	0	− 0.3244	0	+0.8192	0	+0.6159	+0.2000	0	0	− 2.045	0	0	+1.000
$\phi_k=40$:																
40	− 1.764	0	−2.398	−0.3556	− 0.3096	0	+0.9087	+0.1962	−0.1273	0	−0.0525	−0.0035	− 1.421	0	− 0.3885	+0.8156
30	− 1.640	−0.947	−2.105	−0.3239	− 0.1336	−0.1389	+0.9351	+0.1845	−0.0880	−0.0626	−0.0335	−0.0029	− 0.631	− 0.6461	− 0.2007	+0.8306
20	− 0.965	−1.697	−1.251	−0.2374	+ 0.2243	−0.1133	+0.9680	+0.1487	+0.0323	−0.0817	+0.0109	−0.0013	+ 1.015	− 0.5396	+ 0.1574	+0.8707
10	+ 1.111	−1.753	−0.109	−0.1188	+ 0.2753	+0.0557	+0.9101	+0.0870	+0.2408	−0.0111	+0.0434	−0.0007	+ 1.316	+ 0.2523	+ 0.2250	+0.9271
0	+ 5.836	0	+0.643	0	− 0.7210	0	+0.7660	0	+0.5460	+0.2000	0	0	− 3.452	0	0	+1.000
$\phi_k=45$:																
45	− 0.775	0	−1,934	−0.3619	− 0.5679	0	+0.7941	+0.2293	−0.1067	0	−0.0577	−0.0050	− 2.074	0	− 0.7365	+0.7509
40	− 0.820	−0.217	−1.897	−0.3556	− 0.4987	−0.1493	+0.8118	+0.2269	−0.0988	−0.0287	−0.0535	−0.0048	− 1.828	− 0.5460	− 0.6582	+0.7558
30	− 1.062	−0.731	−1.584	−0.3068	− 0.0210	−0.3078	+0.9229	+0.2071	−0.0506	−0.0726	−0.0228	−0.0034	− 0.116	− 1.138	− 0.1410	+0.7920
20	− 0.934	−1.317	−0.910	−0.2175	+ 0.5381	−0.1552	+1.008	+0.1641	+0.0570	−0.0737	+0.0224	−0.0014	+ 1.958	− 0.5987	+ 0.3956	+0.8510
10	+ 0.700	−1.478	+0.039	−0.1073	+ 0.4245	+0.1611	+0.9154	+0.0946	+0.2353	+0.0028	+0.0482	0	+ 1.641	+ 0.5811	+ 0.3778	+0.9196
0	+ 5.344	0	+0.707	0	− 1.392	0	+0.7071	0	+0.4995	+0.2000	0	0	− 5.299	0	0	+1.000
$\phi_k=50$:																
50	+ 0.052	0	−1.437	−0.3493	− 0.9409	0	+0.5813	+0.2514	−0.0848	0	−0.0601	−0.0065	− 2.814	0	− 1.245	+0.6613
40	− 0.291	−0.028	−1.413	−0.3315	− 0.5827	−0.4585	+0.7040	+0.2452	−0.0714	−0.0439	−0.0468	−0.0056	− 1.762	− 1.381	− 0.8336	+0.6869
30	− 1.016	−0.383	−1.246	−0.2786	+ 0.2885	−0.5526	+0.9656	+0.2228	−0.0258	−0.0723	−0.0113	−0.0035	+ 0.797	− 1.690	+ 0.0859	+0.7523
20	− 1.244	−1.059	−0.745	−0.1953	+ 0.1015	−0.1620	+1.112	+0.1764	+0.0664	−0.0639	+0.0312	−0.0013	+ 3.038	− 0.536	+ 0.7607	+0.8338
10	+ 0.381	−1.427	+0.110	−0.0943	+ 0.5624	+0.3610	+0.9471	+0.1015	+0.2257	+0.0125	+0.0513	0	+ 1.811	+ 1.083	+ 0.5718	+0.9154
0	+ 5.670	0	+0.766	0	− 2.398	0	+0.6428	0	+0.4748	+0.2000	0	0	− 7.473	0	0	+1.000
(d) $r/t = 100$ and $r/l = 0.3$																
$\phi_k=30$:																
30	− 4.709	0	−3.362	−0.2769	− 0.2662	0	+0.9571	+0.1099	−0.2618	0	−0.0596	−0.0024	− 2.101	0	− 0.3444	+0.7859
20	− 3.568	−2.381	−2.632	−0.2347	− 0.0094	−0.0961	+0.9752	+0.0982	−0.1182	−0.1174	−0.0231	−0.0016	− 0.108	− 0.7669	− 0.0802	+0.8119
10	+ 1.053	−3.290	−0.837	−0.1275	+ 0.2942	+0.0035	+0.9639	+0.0622	+0.3251	−0.0750	+0.0421	−0.0003	+ 2.372	+ 0.0141	+ 0.1945	+0.8845
0	+12.43	0	+0.500	0	− 0.6606	0	+0.8660	0	+1.102	+0.3000	0	0	− 5.456	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	Vertical Edge Load				Horizontal Edge Load				Shear Edge Load				Edge Moment Load			
	T_x	S	T_ϕ	M_ϕ	T_x	S	T_ϕ	M_ϕ	T_x	S	T_ϕ	M_ϕ	T_x	S	T_ϕ	M_ϕ
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
								(d) (Continued)								
$\phi_k = 35$:																
35	− 2.156	0	−2.682	−0.2903	− 0.6039	0	+0.8706	+0.1362	−0.2140	0	−0.0685	−0.0036	− 3.493	0	− 0.7613	+0.7084
30	− 2.203	−0.595	−2.591	−0.2821	− 0.4823	−0.1544	+0.8892	+0.1338	−0.1937	−0.0568	−0.0603	−0.0033	− 2.808	− 0.8945	− 0.6319	+0.7161
20	− 2.106	−1.817	−1.850	−0.2201	+ 0.2698	−0.2300	+0.9859	+0.1137	−0.0244	−0.1248	−0.0046	−0.0018	+ 1.506	− 1.358	+ 0.1098	+0.7726
10	+ 0.513	−2.456	−0.482	−0.1148	+ 0.6242	+0.0755	+0.9800	+0.0698	+0.3456	−0.0471	+0.0535	−0.0002	+ 3.764	+ 0.4236	+ 0.4372	+0.8689
0	+10.03	0	+0.574	0	− 1.543	0	+0.8192	0	+0.9676	+0.3000	0	0	− 9.403	0	0	+1.000
$\phi_k = 40$:																
40	− 0.216	0	−1.977	−0.2853	− 1.176	0	+0.6733	+0.1553	−0.1668	0	−0.0736	−0.0048	− 5.193	0	− 1.469	+0.6071
30	− 1.020	−0.275	−1.892	−0.2631	− 0.4865	−0.5149	+0.8208	+0.1488	−0.1238	−0.0837	−0.0485	−0.0039	− 2.222	− 2.288	− 0.7475	+0.6444
20	− 2.210	−1.203	−1.430	−0.1982	+ 0.9141	−0.3958	+1.069	+0.1252	+0.0263	−0.1164	+0.0125	−0.0018	+ 3.993	− 1.809	+ 0.5716	+0.7382
10	− 0.295	−2.141	−0.338	−0.1018	+ 1.073	+0.2676	+1.040	+0.0762	+0.3397	−0.0252	+0.0611	−0.0001	+ 5.004	+ 1.180	+ 0.7966	+0.8572
0	+10.01	0	+0.643	0	− 3.068	0	+0.7660	0	+0.8034	+0.3000	0	0	− 14.38	0	0	+1.000
$\phi_k = 45$:																
45	+ 1.400	0	−1.178	−0.2565	− 1.970	0	+0.3055	+0.1591	−0.1142	0	+0.0719	−0.0059	− 6.814	0	− 2.463	+0.4746
40	+ 1.083	+0.354	−1.224	−0.2537	− 1.728	−0.5178	+0.3766	+0.1589	−0.1112	−0.0310	−0.0674	−0.0056	− 6.032	− 1.802	− 2.202	+0.4871
30	− 1.040	+0.432	−1.454	−0.2296	− 0.0472	−1.061	+0.8385	+0.1550	−0.0776	−0.0851	−0.0328	−0.0040	− 0.316	− 3.743	− 0.4779	+0.5768
20	− 3.110	−0.788	−1.326	−0.1743	+ 1.941	−0.5051	+1.279	+0.1338	+0.0382	−0.1013	+0.0241	−0.0016	+ 6.770	− 1.867	+ 1.312	+0.7124
10	− 0.946	−2.214	−0.330	−0.0899	+ 1.513	+0.6328	+1.158	+0.0821	+0.3209	−0.0128	+0.0645	−0.0001	+ 5.696	+ 2.230	+ 1.229	+0.8510
0	+11.24	0	+0.707	0	− 5.257	0	+0.7071	0	+0.8743	+0.3000	0	0	− 19.61	0	0	+1.000
$\phi_k = 50$:																
50	+ 2.605	0	−0.326	−0.2300	− 2.821	0	−0.2407	+0.1409	−0.0565	0	−0.0611	−0.0063	− 7.871	0	− 3.553	+0.3153
40	+ 1.276	+1.180	−0.676	−0.2032	− 1.744	−1.346	+0.1577	+0.1474	−0.0651	−0.0327	−0.0514	−0.0056	− 4.994	− 3.782	− 2.394	+0.3754
30	− 1.837	+1.062	−1.342	−0.1926	+ 0.8732	−1.624	+1.033	+0.1561	−0.0643	−0.0700	−0.0208	−0.0036	+ 2.207	− 4.666	+ 0.2204	+0.5257
20	− 4.099	−0.690	−1.432	−0.1527	+ 3.129	−0.4486	+1.623	+0.1422	+0.0214	−0.0879	+0.0276	−0.0013	+ 8.917	− 1.422	+ 2.171	+0.6995
10	− 1.187	−2.497	−0.379	−0.0801	+ 1.766	+1.155	+1.315	+0.0885	+0.3005	−0.0113	+0.0639	+0.0002	+ 5.588	+ 3.317	+ 1.625	+0.8508
0	+12.66	0	+0.766	0	− 7.773	0	+0.6428	0	+0.8992	+0.3000	0	0	− 23.76	0	0	+1.000
								(e) $r/t = 100$ and $r/l = 0.4$								
$\phi_k = 30$:																
30	− 3.508	0	−3.175	−0.2404	− 0.736	0	+0.881	+0.0947	−0.3318	0	−0.0772	−0.0028	− 5.33	0	− 0.879	+0.6558
20	− 3.500	−1.942	−2.592	−0.2048	− 0.038	−0.2682	+0.958	+0.0856	−0.1568	−0.1503	−0.0304	−0.0019	− 0.14	− 1.920	− 0.196	+0.6995
10	− 0.244	−3.281	−0.947	−0.1121	+ 0.804	−0.0008	+1.008	+0.0544	+0.4153	−0.0998	+0.0545	−0.0003	+ 6.27	+ 0.187	+ 0.502	+0.8169
0	+15.23	0	+0.500	0	− 1.772	0	+0.866	0	+1.508	+0.4000	0	0	− 15.27	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	Vertical Edge Load				Horizontal Edge Load				Shear Edge Load				Edge Moment Load			
	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
(e) (Continued)																
$\phi_k=35$:																
35	− 0.217	0	−2.261	−0.2351	− 1.518	0	+0.670	+0.1091	−0.2518	0	−0.0842	−0.0040	− 8.48	0	− 1.890	+0.5364
30	− 0.656	−0.102	−2.241	−0.2293	− 1.214	−0.3878	+0.723	+0.1078	−0.2316	−0.0672	−0.0746	−0.0037	− 6.82	− 2.173	− 1.574	+0.5500
20	− 2.982	−1.086	−1.911	−0.1833	+ 0.680	−0.5777	+1.014	+0.0952	−0.0480	−0.1540	−0.0073	−0.0019	+ 3.69	− 3.300	+ 0.248	+0.6463
10	− 1.512	−2.715	−0.712	−0.0982	+ 1.596	+0.1982	+1.092	+0.0612	+0.4268	−0.0671	+0.0673	−0.0001	+ 9.37	+ 1.106	+ 1.064	+0.8000
0	+15.11	0	+0.574	0	− 3.990	0	+0.819	0	+1.373	+0.4000	0	0	− 23.99	0	0	+1.000
$\phi_k=40$:																
40	+ 2.485	0	−1.221	−0.2044	− 2.664	0	+0.250	+0.1088	−0.1542	0	−0.0818	−0.0049	− 11.15	0	− 3.211	+0.3896
30	+ 0.117	+0.913	−1.512	−0.1948	− 1.123	−1.171	+0.606	+0.1093	−0.1315	−0.0851	−0.0567	−0.0039	− 4.83	− 4.928	− 0.164	+0.4541
20	− 4.296	−0.278	−1.730	−0.1580	+ 2.058	−0.9139	+1.236	+0.1013	+0.0023	−0.1351	+0.0099	−0.0017	+ 8.71	− 3.904	+ 1.250	+0.6098
10	− 2.814	−2.749	−0.749	−0.0862	+ 2.495	+0.6036	+1.272	+0.0665	+0.4228	−0.0471	+0.0728	0	+ 11.30	+ 2.733	+ 1.764	+0.7918
0	+17.20	0	+0.643	0	− 7.122	0	+0.766	0	+1.368	+0.4000	0	0	− 33.39	0	0	+1.000
$\phi_k=45$:																
45	+ 4.329	0	−0.119	−0.1513	− 3.817	0	−0.372	+0.0899	−0.0653	0	−0.0657	−0.0051	− 12.50	0	− 4.647	+0.2210
40	+ 3.663	+1.126	−0.277	−0.1526	− 3.361	−1.005	−0.230	+0.0923	−0.0723	−0.0186	−0.0630	−0.0049	− 11.09	− 3.299	− 4.168	+0.2412
30	− 0.918	+2.029	−1.249	−0.1564	− 0.150	−2.079	+0.703	+0.1060	−0.1030	−0.0673	−0.0384	−0.0035	− 0.92	− 6.970	− 0.971	+0.3846
20	− 6.034	−0.008	−1.889	−0.1377	+ 3.796	−1.015	+1.638	+0.1083	−0.0344	−0.1142	+0.0158	−0.0013	+ 12.56	− 3.637	+ 2.441	+0.5941
10	− 3.355	−3.194	−0.854	−0.0780	+ 3.104	+1.260	+1.497	+0.0731	+0.3651	−0.0449	+0.0716	+0.0003	+ 11.51	+ 4.262	+ 2.363	+0.7936
0	+19.64	0	+0.707	0	−10.78	0	+0.707	0	+1.388	+0.4000	0	0	− 39.59	0	0	+1.000
$\phi_k=50$:																
50	+ 4.973	0	+0.784	−0.0881	− 4.477	0	−1.043	+0.0545	+0.0216	0	−0.0395	−0.0045	− 11.45	0	− 5.493	+0.0676
40	+ 2.829	+2.328	+0.079	−0.1028	− 2.858	−2.155	−0.394	+0.0716	−0.0276	+0.0025	−0.0408	−0.0041	− 7.65	− 4.182	− 3.773	+0.1539
30	− 2.433	+2.516	−1.399	−0.1279	+ 1.235	−2.673	+1.065	+0.1057	−0.1165	−0.0382	−0.0313	−0.0029	+ 2.61	− 3.745	+ 0.216	+0.3594
20	− 6.881	−0.213	−2.118	−0.1244	+ 5.125	−0.8202	+2.120	+0.1184	−0.0779	−0.1025	+0.0109	−0.0010	+ 13.82	− 1.157	+ 3.391	+0.5981
10	− 3.049	−3.540	−0.904	−0.0724	+ 3.212	+1.909	+1.707	+0.0810	+0.3428	−0.0541	+0.0667	+0.0003	+ 10.19	+ 5.359	+ 2.668	+0.7998
0	+20.35	0	+0.766	0	−13.59	0	+0.643	0	+1.459	+0.4000	0	0	− 41.80	0	0	+1.000
(f) − r/t = 100 and r/l = 0.5																
$\phi_k=30$:																
30	− 1.757	0	−2.879	−0.2019	− 1.420	0	+0.767	+0.0789	−0.3849	0	−0.0912	−0.0030	− 10.64	0	− 1.779	+0.5185
20	− 3.508	−1.323	−2.524	−0.1738	− 0.037	−0.5108	+0.932	+0.0727	−0.1963	−0.1773	−0.0368	−0.0020	− 0.48	− 3.877	− 0.421	+0.5825
10	− 2.292	−3.378	−1.108	−0.0969	+ 1.616	+0.0381	+1.071	+0.0489	+0.4854	−0.1263	+0.0654	−0.0002	+ 12.53	+ 0.236	+ 0.998	+0.7480
0	+20.22	0	+0.500	0	− 3.767	0	+0.866	0	+1.964	+0.5000	0	0	− 30.35	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	Vertical Edge Load				Horizontal Edge Load				Shear Edge Load				Edge Moment Load			
	T_x	S	T_ϕ	M_ϕ	T_x	S	T_ϕ	M_ϕ	T_x	S	T_ϕ	M_ϕ	T_x	S	T_ϕ	M_ϕ
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
(f) (Continued)																
$\phi_k=35$:																
35	+ 2.471	0	−1.677	−0.1788	− 2.776	0	+0.395	+0.0816	−0.2558	0	−0.0962	−0.0039	− 14.99	0	− 3.345	+0.3692
30	+ 1.493	+0.587	−1.758	−0.1758	− 2.223	−0.7101	+0.495	+0.0816	−0.2428	−0.0837	−0.0875	−0.0037	− 12.12	− 4.668	− 2.786	+0.3888
20	− 4.209	−0.060	−2.002	−0.1480	+ 1.246	−1.062	+1.056	+0.0775	−0.0879	−0.1706	−0.0227	−0.0018	+ 6.40	− 5.907	+ 0.453	+0.5266
10	− 4.426	−3.090	−1.055	−0.0837	+ 2.986	+0.3737	+1.252	+0.0535	+0.4733	−0.0921	+0.0629	+0.0000	+ 17.12	+ 1.978	+ 1.931	+0.7345
0	+22.54	0	+0.574	0	− 7.555	0	+0.819	0	+1.867	+0.5000	0	0	− 44.56	0	0	+1.000
$\phi_k=40$:																
40	+ 5.472	0	−0.358	−0.1324	− 4.282	0	−0.220	+0.0682	−0.1143	0	−0.0775	−0.0034	− 17.17	0	− 5.060	+0.2033
30	+ 1.404	+2.232	−1.069	−0.1354	− 1.839	−1.889	+0.363	+0.0754	−0.1408	−0.0683	−0.0585	−0.0034	− 7.86	− 7.678	− 2.645	+0.2912
20	− 6.661	+0.757	−2.062	−0.1254	+ 3.336	−1.488	+1.415	+0.0819	−0.0853	−0.1408	+0.0027	−0.0014	+ 13.14	− 6.372	+ 1.904	+0.5001
10	− 5.826	−3.503	−1.214	−0.0751	+ 4.196	+1.017	+1.528	+0.0593	+0.4218	−0.0802	+0.0780	+0.0003	+ 18.72	+ 4.129	+ 2.821	+0.7347
0	+26.12	0	+0.643	0	−12.17	0	+0.766	0	+1.908	+0.5000	0	0	− 54.72	0	0	+1.000
$\phi_k=45$:																
45	+ 6.655	0	+0.771	−0.0750	− 4.823	0	−0.923	+0.0405	+0.0153	0	−0.0477	−0.0038	− 15.73	0	− 6.128	+0.0523
40	+ 5.749	+1.741	+0.524	−0.0793	− 4.625	−1.375	−0.727	+0.0448	−0.0054	+0.0023	−0.0480	−0.0037	− 14.17	− 4.167	− 5.525	+0.0765
30	− 0.652	+3.364	−1.053	−0.1039	− 0.365	−2.902	+0.573	+0.0709	−0.1266	−0.0318	−0.0417	−0.0028	− 2.11	− 9.122	− 1.418	+0.2487
20	− 8.443	+0.730	−2.362	−0.1125	+ 5.266	−1.497	+1.929	+0.0901	−0.1395	−0.1180	+0.0012	−0.0011	+ 16.12	− 5.253	+ 3.200	+0.5015
10	− 5.758	−4.033	−1.326	−0.0708	+ 4.705	+1.781	+1.798	+0.0670	+0.3807	−0.0882	+0.0726	+0.0003	+ 17.30	+ 5.658	+ 3.294	+0.7426
0	+27.76	0	+0.707	0	−16.12	0	+0.707	0	+2.002	+0.5000	0	0	− 58.57	0	0	+1.000
$\phi_k=50$:																
50	+ 6.074	0	+1.416	−0.0253	− 5.122	0	−1.462	+0.0088	+0.1028	0	−0.0139	−0.0027	− 11.38	0	− 6.094	−0.0520
40	+ 3.738	+2.900	+0.542	−0.0469	− 3.478	−2.507	−0.713	+0.0302	+0.0147	+0.0398	−0.0269	−0.0027	− 8.59	− 5.742	− 4.363	+0.0339
30	− 2.427	+3.394	−1.371	−0.0900	+ 1.103	−3.275	+1.025	+0.0759	−0.1664	−0.0015	−0.0410	−0.0022	+ 0.92	− 8.235	− 0.093	+0.2533
20	− 8.536	+0.212	−2.512	−0.1065	+ 6.245	−1.164	+2.385	+0.1024	−0.1868	−0.1144	−0.0079	−0.0010	+ 15.24	− 3.756	+ 3.776	+0.5154
10	− 4.760	−4.221	−1.261	−0.0672	+ 4.596	+2.387	+1.966	+0.0752	+0.3719	−0.1018	+0.0671	+0.0003	+ 14.96	+ 6.079	+ 3.305	+0.7497
0	+26.68	0	+0.766	0	−18.51	0	+0.643	0	+2.065	+0.5000	0	0	− 56.51	0	0	+1.000
(g) $r/t = 100$ and $r/l = 0.6$																
$\phi_k=30$:																
30	+ 0.690	0	−2.477	−0.1646	− 2.375	0	+0.609	+0.0636	−0.4130	0	−0.1015	−0.0031	− 17.24	0	− 2.909	+0.3912
20	− 3.149	−0.438	−2.432	−0.1447	− 0.079	−0.8580	+0.895	+0.0605	−0.2349	−0.1448	−0.0424	−0.0010	− 0.99	− 6.327	− 0.707	+0.4746
10	− 5.088	−3.429	−1.334	−0.0837	+ 2.720	+0.0560	+1.160	+0.0431	+0.5284	−0.1530	+0.0739	−0.0001	+ 16.89	+ 0.297	+ 1.631	+0.6844
0	+26.76	0	+0.500	0	− 6.379	0	+0.866	0	+2.481	+0.6000	0	0	− 50.48	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	Vertical Edge Load				Horizontal Edge Load				Shear Edge Load				Edge Moment Load			
	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
								(g) (Continued)								
$\phi_k=35$:																
35	+ 5.530	0	+0.999	−0.1294	− 4.187	0	+0.085	+0.0577	−0.2245	0	−0.0937	−0.0036	− 21.54	0	− 4.900	+0.2286
30	+ 3.964	+1.371	−1.194	−0.1291	− 3.368	−1.072	+0.231	+0.0589	−0.2258	−0.0617	−0.0849	−0.0034	− 17.56	− 5.539	− 4.097	+0.2534
20	− 5.521	+1.142	−2.096	−0.1187	+ 1.835	−1.625	+1.098	+0.0628	−0.1434	−0.1738	−0.0163	−0.0014	+ 8.78	− 8.709	+ 0.609	+0.4257
10	− 7.862	−3.492	−1.452	−0.0731	+ 4.617	+0.5579	+1.438	+0.0477	+0.4825	−0.1215	+0.0821	+0.0005	+ 25.70	+ 2.778	+ 2.864	+0.6789
0	+31.29	0	+0.574	0	−11.74	0	+0.891	0	+2.453	+0.6000	0	0	− 67.87	0	0	+1.000
$\phi_k=40$:																
40	+ 8.029	0	+0.412	−0.0781	− 5.611	0	+0.632	+0.0378	−0.0378	0	−0.0653	−0.0034	− 20.91	0	− 6.413	+0.0741
30	+ 2.640	+3.389	−0.660	−0.0907	− 2.518	−2.498	+0.140	+0.0500	−0.1275	−0.0385	−0.0561	−0.0020	− 10.34	− 9.508	− 3.449	+0.1748
20	− 8.598	+1.754	−2.347	−0.1015	+ 4.328	−2.036	+1.563	+0.0673	−0.1799	−0.1354	−0.0078	−0.0012	+ 16.02	− 8.383	+ 2.316	+0.4147
10	− 8.783	−4.118	−1.651	−0.0674	+ 5.859	+1.341	+1.421	+0.0538	+0.4136	−0.1192	+0.0787	+0.0004	+ 25.53	+ 5.071	+ 3.686	+0.6842
0	+34.71	0	+0.643	0	−17.02	0	+0.766	0	+2.558	+0.6000	0	0	− 75.81	0	0	+1.000
$\phi_k=45$:																
45	+ 7.904	0	+1.346	−0.0293	− 5.855	0	−1.258	+0.0115	+0.1009	0	−0.0267	−0.0026	− 15.56	0	− 6.655	−0.0364
40	+ 6.935	+2.078	+1.051	−0.0352	− 5.259	−1.550	−1.038	+0.0166	+0.0669	+0.0245	−0.0302	−0.0025	− 14.36	− 4.156	− 6.057	−0.0122
30	− 0.196	+4.195	−0.884	−0.0712	− 0.727	−3.373	+0.459	+0.0490	−0.1460	+0.0082	−0.0439	−0.0022	− 3.96	− 9.685	− 1.817	+0.1633
20	− 9.841	+1.352	−2.657	−0.0959	+ 6.025	−1.905	+2.094	+0.0775	−0.2510	−0.1174	−0.0148	−0.0010	+ 16.43	− 6.537	+ 3.383	+0.4285
10	− 7.893	−4.563	−1.682	−0.0659	+ 6.150	+2.089	+2.020	+0.0623	+0.3813	−0.1339	+0.0714	+0.0004	+ 22.53	+ 6.087	+ 3.873	+0.6947
0	+34.56	0	+0.707	0	−20.64	0	+0.707	0	+2.659	+0.6000	0	0	− 74.68	0	0	+1.000
$\phi_k=50$:																
50	+ 6.029	0	+1.647	+0.0038	− 4.847	0	−1.571	−0.0114	+0.1678	0	+0.0073	−0.0015	− 8.26	0	− 5.666	−0.0950
40	+ 4.096	+2.955	+0.283	−0.0192	− 3.656	−2.446	−0.851	+0.0100	+0.0523	+0.0699	−0.0151	−0.0017	− 8.00	− 4.483	− 4.344	−0.0164
30	− 1.803	+3.754	−1.264	−0.0627	+ 0.458	−3.469	+0.919	+0.0574	−0.2030	+0.0305	−0.0491	−0.0018	− 1.93	− 7.812	− 0.623	+0.1835
20	− 9.145	+0.659	−2.736	−0.0932	+ 6.530	−1.539	+2.469	+0.0897	−0.2862	−0.1225	−0.0245	−0.0010	+ 13.86	− 5.044	+ 3.549	+0.4425
10	− 6.445	−4.528	−1.501	−0.0624	+ 6.010	+2.570	+2.141	+0.0696	+0.3899	−0.1463	+0.0675	+0.0003	+ 20.05	+ 5.942	+ 3.658	+0.6986
0	+31.83	0	+0.766	0	−22.64	0	+0.643	0	+2.688	+0.6000	0	0	− 70.37	0	0	+1.000
								(h) $r/t = 200$ and $r/l = 0.1$								
$\phi_k=30$:																
30	− 5.388	0	−3.482	−0.3278	− 0.0129	0	+0.9979	+0.1310	−0.0905	0	−0.0207	−0.0009	− 0.1005	0	− 0.0164	+0.9730
20	− 3.600	−2.628	−2.666	−0.2779	+ 0.0014	−0.0043	+0.9845	+0.1162	−0.0397	−0.0403	−0.0080	−0.0006	+ 0.0123	− 0.0332	− 0.0024	+0.9761
10	+ 1.737	−3.290	−0.781	−0.1511	+ 0.0164	+0.0020	+0.9410	+0.0722	+0.1117	−0.0251	+0.0142	−0.0002	+ 0.1340	+ 0.0177	+ 0.0099	+0.9852
0	+10.83	0	+0.500	0	− 0.0447	0	+0.8660	0	+0.3604	+0.1000	0	0	− 0.3807	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	VERTICAL EDGE LOAD				HORIZONTAL EDGE LOAD				SHEAR EDGE LOAD				EDGE MOMENT LOAD			
	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
(*i*) (Continued)																
$\phi_k=35$:																
35	−2.269	0	−2.720	−0.3304	−0.5645	0	+0.8832	+0.1560	−0.1455	0	−0.0466	−0.0027	−3.344	0	−0.7035	+0.8373
30	−2.290	−0.624	−2.624	−0.3211	−0.4523	−0.1446	+0.9000	+0.1531	−0.1313	−0.0386	−0.0410	−0.0025	−2.696	−0.8582	−0.5812	+0.8425
20	−2.035	−1.854	−1.855	−0.2507	+0.2416	−0.2183	+0.9868	+0.1291	−0.0150	−0.0842	−0.0033	−0.0014	+1.373	−1.319	+0.1200	+0.8787
10	+0.648	−2.423	−0.474	−0.1313	+0.5770	+0.0604	+0.9767	+0.0783	+0.2331	−0.0309	+0.0358	−0.0002	+3.519	+0.3350	+0.4303	+0.9335
0	+9.620	0	+0.574	0	−1.373	0	+0.8191	0	+0.6376	+0.2000	0	0	−8.444	0	0	+1.000
$\phi_k=40$:																
40	−0.324	0	−2.021	−0.3363	−1.133	0	+0.6921	+0.1847	−0.1144	0	−0.0503	−0.0037	−5.147	0	−1.422	+0.7549
30	−1.057	−0.319	−1.919	−0.3098	−0.4724	−0.4965	+0.8324	+0.1760	−0.0838	−0.0572	−0.0331	−0.0030	−2.213	−2.270	−0.7127	+0.7838
20	−2.100	−1.229	−1.425	−0.2331	+0.8653	−0.3869	+1.067	+0.1458	+0.0202	−0.0785	+0.0084	−0.0015	+3.880	−1.816	+0.5816	+0.8519
10	−0.178	−2.090	−0.327	−0.1202	+1.021	+0.2424	+1.036	+0.0873	+0.2295	−0.0158	+0.0411	−0.0002	+4.849	+1.085	+0.8003	+0.9255
0	+9.537	0	+0.643	0	−2.851	0	+0.7660	0	+0.5858	+0.2000	0	0	−13.55	0	0	+1.000
$\phi_k=45$:																
45	+1.358	0	−1.210	−0.3140	−1.967	0	+0.3169	+0.1973	−0.0793	0	−0.0494	−0.0047	−7.076	0	−2.500	+0.6266
40	+1.046	+0.344	−1.254	−0.3103	−1.725	−0.5171	+0.3877	+0.1967	−0.0769	−0.0215	−0.0462	−0.0045	−6.231	−1.862	−2.231	+0.6378
30	−1.033	−0.270	−1.469	−0.2789	−0.050	−0.6113	+0.8459	+0.1884	−0.0516	−0.0642	−0.0223	−0.0032	−0.316	−2.536	−0.5827	+0.7167
20	−3.035	−0.788	−1.324	−0.2104	+1.916	−0.5110	+1.281	+0.1588	+0.0289	−0.0681	+0.0165	−0.0014	+6.897	−1.941	+1.367	+0.8267
10	−0.885	−2.169	−0.327	−0.1087	+1.486	+0.6094	+1.159	+0.0955	+0.2164	−0.0071	+0.0436	−0.0001	+5.708	+2.201	+1.273	+0.9200
0	+10.91	0	+0.707	0	−5.085	0	+0.7071	0	+0.5724	+0.2000	0	0	−19.32	0	0	+1.000
$\phi_k=50$:																
50	+2.686	0	−0.312	−0.2588	−2.918	0	−0.2665	+0.1807	−0.0395	0	−0.0419	−0.0052	−8.502	0	−3.770	+0.4461
40	+1.320	+1.218	−0.671	−0.2562	−1.798	−1.391	+0.1445	+0.1856	−0.0445	−0.0226	−0.0351	−0.0046	−5.350	−4.060	−2.523	+0.5106
30	−1.860	+1.105	−1.355	−0.2385	+0.904	−1.675	+1.046	+0.1889	−0.0420	−0.0476	−0.0140	−0.0031	+2.433	−4.983	+0.2750	+0.6671
20	−4.145	−0.669	−1.455	−0.1867	+3.198	−0.4671	+1.649	+0.1655	+0.0168	−0.0585	+0.0187	−0.0013	+9.459	−1.527	+2.336	+0.8314
10	−1.190	−2.490	−0.395	−0.0979	−1.779	+1.160	+1.332	+0.0992	+0.2016	−0.0062	+0.0428	−0.0001	+5.734	+3.392	+1.737	+0.9393
0	+12.58	0	+0.766	0	−7.777	0	+0.6428	0	+0.5914	+0.2000	0	0	−24.30	0	0	+1.000
(*j*) $r/t = 200$ and $r/l = 0.3$																
$\phi_k=30$:																
30	−2.875	0	−3.097	−0.2688	−1.018	0	+0.8433	+0.1065	−0.2481	0	−0.0585	−0.0023	−8.01	0	−1.251	+0.7583
20	−3.544	−1.727	−2.595	−0.2300	−0.025	−0.3675	+0.9558	+0.0961	−0.1187	−0.0759	−0.0236	−0.0016	−0.32	−2.906	−0.242	+0.7953
10	−1.006	−3.351	−1.021	−0.1270	+1.142	+0.0251	+1.038	+0.0619	+0.3098	−0.1126	+0.0402	−0.0003	+9.20	+0.158	+0.787	+0.8828
0	+17.18	0	+0.500	0	−2.618	0	+0.8660	0	+1.138	+0.3000	0	0	−21.66	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	Vertical Edge Load				Horizontal Edge Load				Shear Edge Load				Edge Moment Load			
	T_x	S	T_ϕ	M_ϕ	T_x	S	T_ϕ	M_ϕ	T_x	S	T_ϕ	M_ϕ	T_x	S	T_ϕ	M_ϕ
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
(*j*) (Continued)																
$\phi_k = 35$:																
35	+ 1.157	0	−2.003	−0.2639	− 3.540	0	+0.5377	+0.1233	−0.1777	0	−0.0616	−0.0033	− 12.73	0	− 2.708	+0.6320
30	+ 0.434	+0.250	−2.034	−0.2580	− 1.763	−0.5643	+0.6157	+0.1220	−0.1650	−0.0475	−0.0548	−0.0031	− 10.23	− 3.260	− 2.236	+0.6464
20	− 3.630	−0.571	−1.989	−0.2098	+ 1.001	−0.8379	+1.051	+0.1087	−0.0418	−0.1115	−0.0066	−0.0017	+ 5.57	− 4.934	+ 0.474	+0.7427
10	− 2.937	−2.907	−0.903	−0.1149	+ 2.318	+0.2927	+1.184	+0.0698	+0.3091	−0.0522	+0.0487	−0.0002	+ 13.92	+ 1.650	+ 1.650	+0.8689
0	+18.66	0	+0.574	0	− 5.780	0	+0.8192	0	+1.058	+0.3000	0	0	− 35.32	0	0	+1.000
$\phi_k = 40$:																
40	+ 4.491	0	−0.695	−0.2230	− 4.689	0	−0.0553	+0.1198	−0.0958	0	−0.0550	−0.0039	− 16.80	0	− 4.723	+0.4470
30	+ 0.927	+1.787	−1.263	−0.2170	− 1.599	−1.680	+0.4612	+0.1225	−0.0958	−0.0529	−0.0397	−0.0032	− 7.30	− 7.430	− 2.380	+0.5253
20	− 5.904	+0.374	−1.979	−0.1836	+ 2.992	−1.296	+1.378	+0.1168	−0.0298	−0.0936	+0.0040	−0.0016	+ 12.97	− 5.929	+ 1.932	+0.6997
10	− 4.643	−3.248	−1.057	−0.1038	+ 3.576	+0.8937	+1.451	+0.0774	+0.2786	−0.0416	+0.0506	−0.0001	+ 16.75	+ 3.932	+ 2.673	+0.8621
0	+22.55	0	+0.643	0	−10.29	0	+0.7660	0	+1.069	+0.3000	0	0	− 48.57	0	0	+1.000
$\phi_k = 45$:																
45	+ 6.463	0	+0.608	−0.1511	− 5.443	0	−0.8614	+0.0899	−0.0114	0	−0.0370	−0.0038	− 17.92	0	− 6.551	+0.2280
40	+ 5.541	+1.688	+0.369	−0.1543	− 4.617	−1.379	−0.6652	+0.0937	−0.0214	−0.0039	−0.0364	−0.0037	− 15.90	− 4.718	− 5.866	+0.2542
30	− 0.843	+3.190	−1.138	−0.1700	− 0.203	−2.859	+0.6257	+0.1158	−0.0772	−0.0304	−0.0273	−0.0028	− 1.34	− 9.994	− 1.292	+0.4354
20	− 8.160	+0.555	−2.324	−0.1610	+ 5.220	−1.401	+1.929	+0.1250	−0.0627	−0.0765	+0.0042	−0.0013	+ 17.97	− 5.233	+ 3.575	+0.6795
10	− 5.055	−3.891	−1.245	−0.0955	+ 4.276	+1.728	+1.762	+0.0858	+0.2474	−0.0456	+0.0471	−0.0000	+ 16.41	+ 6.050	+ 3.437	+0.8640
0	+25.47	0	+0.707	0	−14.77	0	+0.7071	0	+1.134	+0.3000	0	0	− 56.01	0	0	+1.000
$\phi_k = 50$:																
50	+ 6.626	0	+1.510	−0.0742	− 5.740	0	−1.603	+0.0441	+0.0544	0	−0.0135	−0.0031	− 15.39	0	− 7.343	+0.0368
40	+ 3.900	+3.127	+0.564	−0.0969	− 3.696	−2.769	−0.7707	+0.0676	+0.0026	+0.0201	−0.0201	−0.0029	− 10.44	− 7.529	− 5.048	+0.1442
30	− 2.878	+3.516	−1.463	−0.1393	+ 1.552	−3.453	+1.111	+0.1158	−0.1000	−0.0070	−0.0254	−0.0023	+ 3.25	− 9.817	+ 0.304	+0.4060
20	− 8.823	+0.094	−2.598	+0.1467	+ 6.637	−1.072	+2.491	+0.1375	−0.1008	−0.0715	−0.0025	−0.0011	+ 18.60	− 3.576	+ 4.612	+0.6832
10	− 4.273	−4.274	−1.262	−0.0891	+ 4.214	+2.483	+1.987	+0.0953	+0.2354	−0.0563	+0.0424	−0.0000	+ 14.04	+ 7.094	+ 3.666	+0.8700
0	+25.53	0	+0.766	0	−17.75	0	+0.6428	0	+1.193	+0.3000	0	0	− 56.30	0	0	+1.000
(*k*) $r/t = 200$ and $r/l = 0.4$																
$\phi_k = 30$:																
30	+ 1.124	0	−2.479	−0.2211	− 2.625	0	+0.5943	+0.0868	−0.2847	0	−0.0710	−0.0026	− 20.19	0	− 3.173	+0.5913
20	− 3.463	−0.295	−2.481	−0.1937	− 0.066	−0.9430	+0.9092	+0.0810	−0.1583	−0.1339	−0.0302	−0.0018	− 0.87	− 7.343	− 0.619	+0.6619
10	− 5.528	−3.468	−1.407	−0.1117	+ 2.974	+0.0700	+1.194	+0.0553	+0.3622	−0.1029	+0.0493	−0.0003	+ 23.57	+ 0.435	+ 1.999	+0.8155
0	+27.68	0	+0.500	0	− 6.899	0	+0.8660	0	+1.634	+0.4000	0	0	− 56.52	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	Vertical Edge Load				Horizontal Edge Load				Shear Edge Load				Edge Moment Load			
	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
(k) (Continued)																
$\phi_k=35$:																
35	+ 6.873	0	−0.795	−0.1830	− 4.932	0	−0.0417	+0.0837	−0.1462	0	−0.0635	−0.0032	− 27.50	0	− 5.935	+0.3911
30	+ 5.012	+1.712	−1.039	−0.1820	− 3.951	−1.261	+0.1374	+0.0848	−0.1482	−0.0403	−0.0577	−0.0030	− 22.24	− 7.055	− 4.913	+0.4179
20	− 6.203	+1.614	−2.206	−0.1647	+ 2.208	−1.889	+1.153	+0.0862	−0.0986	−0.1152	−0.0124	−0.0017	+ 11.73	−10.85	+ 1.001	+0.5937
10	− 9.087	−3.682	−1.633	−0.1001	+ 5.303	+0.6597	+1.536	+0.0621	+0.3204	−0.0822	+0.0539	−0.0001	+ 31.27	+ 3.583	+ 3.651	+0.8049
0	+34.19	0	+0.574	0	−13.36	0	+0.8192	0	+1.639	+0.4000	0	0	− 80.78	0	0	+1.000
$\phi_k=40$:																
40	+11.34	0	+0.768	−0.1216	− 7.637	0	−0.8602	+0.0620	+0.0319	0	−0.0448	−0.0032	− 32.54	0	− 8.301	+0.2027
30	+ 3.392	+4.247	−0.465	−0.1293	− 2.992	−3.037	+0.0122	+0.0723	−0.0781	−0.0171	−0.0365	−0.0026	− 13.32	−12.80	− 4.292	+0.2888
20	−10.20	+2.361	−2.620	−0.1409	+ 5.333	−2.419	+1.731	+0.0915	−0.1367	−0.0848	−0.0086	−0.0014	+ 21.73	−10.76	+ 3.315	+0.5642
10	−10.34	−4.580	−1.955	−0.0931	+ 6.841	+1.629	+1.957	+0.0704	+0.2620	−0.0845	+0.0492	−0.0000	+ 31.74	+ 6.881	+ 4.858	+0.8083
0	+39.09	0	+0.643	0	−19.78	0	+0.7660	0	+1.744	+0.4000	0	0	− 92.98	0	0	+1.000
$\phi_k=45$:																
45	+ 9.787	0	+1.934	−0.0436	− 7.229	0	−1.692	+0.0199	+0.0957	0	−0.0099	−0.0065	− 22.45	0	− 8.938	−0.0169
40	+ 8.573	+2.572	+1.567	−0.0513	− 6.456	−1.911	−1.419	+0.0264	+0.0691	+0.0238	−0.0133	−0.0043	− 20.42	− 5.970	− 8.074	+0.0165
30	− 0.227	+5.188	−0.825	−0.0982	− 0.709	−4.101	+0.4128	+0.0678	−0.0987	+0.0199	−0.0292	−0.0020	− 4.08	−13.43	− 2.116	+0.2506
20	−11.70	+1.769	−3.045	−0.1296	+ 7.382	−2.215	+2.378	+0.1027	−0.1953	−0.0728	−0.0161	−0.0012	+ 23.35	− 8.260	+ 4.819	+0.5714
10	− 9.060	−5.153	−2.000	−0.0887	+ 7.017	+2.520	+2.255	+0.0800	+0.2396	−0.0981	+0.0429	−0.0000	+ 27.40	+ 8.293	+ 5.115	+0.8168
0	+38.68	0	+0.707	0	−23.69	0	+0.7071	0	+1.826	+0.4000	0	0	− 91.16	0	0	+1.000
$\phi_k=50$:																
50	+ 7.414	0	+2.207	+0.0042	− 6.022	0	−2.055	−0.0134	+0.1395	0	+0.0147	−0.0013	− 13.14	0	− 7.852	−0.1157
40	+ 4.947	+3.616	+1.124	−0.0267	− 4.389	−3.008	−1.161	+0.0151	+0.0505	+0.0598	−0.0042	−0.0015	− 11.30	− 6.932	− 5.801	−0.0061
30	− 2.266	+4.540	−1.356	−0.0910	+ 0.8355	−4.146	+0.9927	+0.0776	−0.1446	+0.0347	−0.0352	−0.0017	− 1.18	−10.95	− 0.390	+0.2725
20	−10.69	+0.849	−3.073	−0.1245	+ 7.857	−1.711	+2.811	+0.1179	−0.2195	−0.0793	−0.0240	−0.0011	+ 19.84	− 6.154	+ 5.079	+0.5892
10	− 7.142	−5.081	−1.774	−0.0835	+ 6.637	+3.045	+2.378	+0.0892	+0.2513	−0.1076	+0.0402	−0.0001	+ 23.56	+ 7.944	+ 4.801	+0.8219
0	+35.00	0	+0.766	0	−25.44	0	+0.6428	0	+1.840	+0.4000	0	0	− 84.42	0	0	+1.000
(l) $r/t = 200$ and $r/l = 0.5$																
$\phi_k=30$:																
30	+ 6.908	0	−1.578	−0.1695	− 4.932	0	+0.2325	+0.0656	−0.2715	0	−0.0759	−0.0026	− 36.86	0	− 5.858	+0.4164
20	− 3.091	+1.791	−2.323	−0.1564	− 2.320	−1.780	+0.8452	+0.0654	−0.1974	−0.1375	−0.0355	−0.0018	− 2.00	−13.49	− 1.174	+0.5258
10	−12.18	−3.626	−1.972	−0.0957	+ 5.663	+0.128	+1.422	+0.0492	+0.3600	−0.1314	+0.05366	−0.0002	+ 43.95	+ 0.71	+ 3.711	+0.7505
0	+43.35	0	+0.500	0	−13.29	0	+0.8660	0	+2.253	+0.5000	0	0	−107.5	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	Vertical Edge Load				Horizontal Edge Load				Shear Edge Load				Edge Moment Load			
	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
								(l) Continued								
$\phi_k=35$:																
35	+12.92	0	+0.520	−0.1106	− 7.760	0	−0.6687	+0.0486	+0.0566	0	−0.0528	−0.0027	− 41.17	0	− 9.181	+0.1851
30	+10.14	+3.193	+0.037	−0.1151	− 6.371	−1.952	−0.3803	+0.0518	−0.0847	−0.0181	−0.0505	−0.0025	− 33.79	−10.62	− 7.611	+0.2222
20	− 8.763	+4.022	−2.425	−0.1266	+ 3.367	−3.036	+1.253	+0.0670	−0.1818	−0.0966	−0.0204	−0.0015	+ 16.53	−16.90	+ 1.429	+0.4668
10	−16.09	−4.484	−2.447	−0.0897	+ 8.692	+1.030	+1.926	+0.0564	+0.2700	−0.1200	+0.0516	−0.0001	+ 50.25	+ 5.26	+ 5.747	+0.7504
0	+52.21	0	+0.574	0	−22.16	0	+0.8192	0	+2.374	+0.5000	0	0	−132.8	0	0	+1.000
$\phi_k=40$:																
40	+13.93	0	+1.963	−0.0443	− 8.898	0	−1.529	+0.0185	+0.1093	0	−0.0175	−0.0020	− 32.96	0	−10.30	−0.0072
30	+ 5.361	+5.888	+0.146	−0.0720	− 4.097	−3.897	−0.3259	+0.0398	−0.0475	+0.0298	−0.0294	−0.0019	− 17.70	−15.29	− 5.586	+0.1389
20	−12.79	+3.938	−3.042	−0.1129	+ 6.622	−3.303	+1.951	+0.0737	−0.2613	−0.0652	−0.0242	−0.0012	+ 24.15	−14.33	+ 3.905	+0.4658
10	−15.31	−5.476	−2.659	−0.0866	+ 9.721	+2.099	+2.350	+0.0645	+0.2136	−0.1319	+0.0433	−0.0000	+ 44.95	+ 8.13	+ 6.422	+0.7611
0	+53.53	0	+0.643	0	−28.17	0	+0.7660	0	+2.509	+0.5000	0	0	−133.2	0	0	+1.000
$\phi_k=45$:																
45	+10.22	0	+2.435	+0.0045	− 7.196	0	−1.959	−0.0106	+0.1830	0	+0.0132	−0.0011	− 18.31	0	− 8.981	−0.1089
40	+ 9.385	+2.771	+2.035	−0.0052	− 6.757	−1.967	−1.675	−0.0034	+0.1446	+0.0466	+0.0066	−0.0012	− 17.78	− 4.98	− 8.263	−0.0771
30	+ 0.957	+5.868	−0.565	−0.0616	− 1.640	−4.438	+0.2286	+0.0431	−0.1100	+0.0637	−0.0303	−0.0014	− 8.90	−13.02	− 2.879	+0.1523
20	−12.61	+2.686	−3.298	−0.1102	+ 7.723	−2.850	+2.505	+0.0881	−0.3155	−0.0676	−0.0341	−0.0011	+ 20.73	−10.69	+ 4.696	+0.4876
10	−12.52	−5.653	−2.457	−0.0832	+ 9.498	+2.791	+2.554	+0.0750	+0.2211	−0.1467	+0.0389	−0.0001	+ 38.31	+ 8.33	+ 6.068	+0.7681
0	+48.96	0	+0.707	0	−30.94	0	+0.7071	0	+2.539	+0.5000	0	0	−123.1	0	0	+1.000
$\phi_k=50$:																
50	+ 5.720	0	+2.151	+0.0279	− 4.333	0	−1.931	−0.0295	+0.1739	0	+0.0295	+0.0002	− 4.66	0	− 6.575	−0.1404
40	+ 5.105	+3.070	+1.265	−0.0015	− 4.431	−2.438	−1.238	−0.0037	+0.0786	+0.0777	+0.0054	−0.0007	− 9.77	− 3.60	− 5.629	−0.0518
30	− 0.025	+4.652	−1.014	−0.0656	− 1.252	−4.161	+0.6846	+0.0569	−0.1578	+0.0593	−0.0387	−0.0013	− 10.31	− 9.96	− 1.682	+0.1945
20	−10.40	+1.897	−3.112	−0.1093	+ 7.442	−2.639	+2.813	+0.1043	−0.2949	−0.0794	−0.0364	−0.0011	+ 16.18	− 9.65	+ 4.675	+0.5157
10	−12.65	−5.039	−2.643	−0.0852	+11.50	+3.115	+3.155	+0.0913	+0.2141	−0.1221	+0.0366	−0.0000	+ 41.78	+ 9.30	+ 7.575	+0.8127
0	+42.10	0	+0.766	0	−31.89	0	+0.6428	0	+2.441	+0.5000	0	0	−116.1	0	0	+1.000
								(m) $r/t = 200$ and $r/l = 0.6$								
$\phi_k=30$:																
30	+13.41	0	−0.550	−0.1215	− 7.494	0	−0.177	+0.0459	−0.2086	0	−0.0732	−0.0024	− 53.95	0	− 8.721	+0.2589
20	− 2.95	+4.174	−2.106	−0.1221	− 0.317	−2.725	+0.756	+0.0512	−0.2355	−0.1233	−0.0394	−0.0016	− 4.01	−20.04	− 1.828	+0.4042
10	−19.88	−3.751	−2.624	−0.0867	+ 8.756	+0.171	+1.682	+0.0443	+0.3019	−0.1613	+0.0532	−0.0001	+ 66.27	+ 0.71	+ 5.559	+0.6031
0	+61.86	0	+0.500	0	−20.84	0	+0.866	0	+3.003	+0.6000	0	0	−166.4	0	0	+1.000

TABLE 2A.—(Continued)

ϕ	Vertical Edge Load				Horizontal Edge Load				Shear Edge Load				Edge Moment Load			
	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
								(m) (Continued)								
$\phi_k=35$:																
35	+17.48	0	+1.581	−0.0576	− 9.794	0	−1.155	+0.0231	+0.0614	0	−0.0363	−0.0020	− 48.68	0	−11.31	+0.0433
30	+13.76	+4.442	+0.942	−0.0648	− 8.004	−2.520	−0.794	+0.0279	+0.0019	+0.0113	−0.0381	−0.0020	− 40.77	−12.62	− 9.488	+0.0867
20	−10.37	+6.000	−2.570	−0.0989	+ 4.010	−3.973	+1.313	+0.0531	−0.2759	−0.0663	−0.0294	−0.0012	+ 17.11	−21.22	+ 1.522	+0.3710
10	−22.33	−5.090	−3.136	−0.0827	+11.70	+1.270	+2.252	+0.0523	+0.1839	−0.1613	+0.0454	−0.0000	+ 66.22	+ 5.83	+ 7.338	+0.7029
0	+68.85	0	+0.574	0	−30.32	0	+0.819	0	+3.204	+0.6000	0	0	−180.9	0	0	+1.000
$\phi_k=40$:																
40	+14.46	0	+2.530	−0.0052	− 8.900	0	−1.788	−0.0029	+0.2096	0	+0.0030	−0.0012	− 29.22	0	−10.53	−0.0850
30	+ 6.70	+6.504	+0.519	−0.0397	− 4.840	−4.143	−0.527	+0.0213	−0.0157	+0.0719	−0.0224	−0.0013	− 20.21	−14.50	− 6.191	+0.0578
20	−13.24	+4.992	−3.187	−0.0946	+ 6.608	−3.889	+2.003	+0.0630	−0.3733	−0.0458	−0.0384	−0.0010	+ 19.37	−16.37	+ 3.594	+0.3909
10	−19.32	−5.810	−3.114	−0.0812	+12.10	+2.231	+2.601	+0.0616	+0.1569	−0.1797	+0.0373	−0.0001	+ 56.09	+ 7.56	+ 7.285	+0.7139
0	+65.23	0	+0.643	0	−35.15	0	+0.766	0	+3.297	+0.6000	0	0	−169.2	0	0	+1.000
$\phi_k=45$:																
45	+ 8.60	0	+2.397	+0.0217	− 5.729	0	−1.858	−0.0205	+0.2347	0	+0.0286	−0.0004	− 8.38	0	− 7.554	−0.1241
40	+ 8.15	+2.319	+2.068	+0.0133	− 5.622	−1.562	−1.636	−0.0140	+0.1924	+0.0605	+0.0199	−0.0005	− 10.38	− 2.49	− 7.198	−0.0988
30	+ 2.68	+5.666	−0.311	−0.0418	− 2.980	−4.165	+0.041	+0.0293	−0.1065	+0.0944	−0.0306	−0.0011	− 15.36	−10.11	− 3.686	+0.0939
20	−11.68	+3.500	−3.237	−0.0948	+ 6.764	−3.438	+2.416	+0.0759	−0.4030	−0.0596	−0.0476	−0.0011	+ 12.42	−13.03	+ 3.667	+0.4113
10	−16.02	−5.641	−2.752	−0.0772	+12.12	+2.709	+2.752	+0.0695	+0.1906	−0.1895	+0.0361	−0.0001	+ 50.54	+ 6.80	+ 6.668	+0.7160
0	+58.53	0	+0.707	0	−38.04	0	+0.707	0	+3.268	+0.6000	0	0	−158.3	0	0	+1.000
$\phi_k=50$:																
50	+ 3.09	0	+1.709	+0.0284	− 1.816	0	−1.467	−0.0273	+0.1854	0	+0.0382	+0.0002	+ 7.18	0	− 3.972	−0.1088
40	+ 4.42	+1.981	+1.179	+0.0063	− 3.693	−1.374	−1.120	−0.0084	+0.1051	+0.0872	+0.0118	−0.0003	− 5.77	+ 1.40	− 4.634	−0.0572
30	+ 2.57	+4.287	−0.638	−0.0476	− 3.650	−3.722	+0.325	+0.0406	−0.1481	+0.0826	−0.0419	−0.0012	− 20.51	− 6.83	− 3.347	+0.1206
20	− 9.53	+2.775	−2.920	−0.0908	+ 6.372	−3.425	+2.581	+0.0862	−0.3819	−0.0757	−0.0497	−0.0012	+ 7.78	−12.83	+ 3.041	+0.4149
20	−14.34	−5.142	−2.383	−0.0706	+13.11	+2.955	+2.877	+0.0757	+0.2199	−0.1883	+0.0384	−0.0001	+ 51.00	+ 5.85	+ 6.371	+0.7134
0	+53.46	0	+0.766	0	−41.80	0	+0.643	0	+3.218	+0.6000	0	0	−156.6	0	0	+1.000

TABLE 2B.—SYMMETRICAL EDGE LOADS ON SIMPLY SUPPORTED CYLINDRICAL SHELLS (DISPLACEMENT OF EDGE AT $\phi = 0$)

	Vertical Edge Load	Horizontal Edge Load	Shear Edge Load	Moment Edge Load
Vertical Displacement Δ_V—(Downward Positive)—	$V_L \frac{l^4}{r^3 t E} \times$ Col. (1) $\times \sin\frac{\pi x}{l}$	$H_L \frac{l^4}{r^3 t E} \times$ Col. (4) $\times \sin\frac{\pi x}{l}$	$S_L \frac{l^4}{r^3 t E} \times$ Col. (7) $\times \sin\frac{\pi x}{l}$	$M_L \times \frac{l^4}{r^4 t E} \times$ Col. (10) $\times \sin\frac{\pi x}{l}$
Horizontal Displacement Δ_H—(Inward Positive)—	$V_L \frac{l^4}{r^3 t E} \times$ Col. (2) $\times \sin\frac{\pi x}{l}$	$H_L \frac{l^4}{r^3 t E} \times$ Col. (5) $\times \sin\frac{\pi x}{l}$	$S_L \frac{l^4}{r^3 t E} \times$ Col. (8) $\times \sin\frac{\pi x}{l}$	$M_L \times \frac{l^4}{r^4 t E} \times$ Col. (11) $\times \sin\frac{\pi x}{l}$
Rotation θ—	$V_L \frac{r^2}{E I} \times$ Col. (3) $\times \sin\frac{\pi x}{l}$	$H_L \frac{r^2}{E I} \times$ Col. (6) $\times \sin\frac{\pi x}{l}$	$S_L \frac{r^2}{E I} \times$ Col. (9) $\times \sin\frac{\pi x}{l}$	$M_L \times \frac{r}{E I} \times$ Col. (12) $\times \sin\frac{\pi x}{l}$

ϕ_k	Δ_V (1)	Δ_H (2)	θ (3)	Δ_V (4)	Δ_H (5)	θ (6)	Δ_V (7)	Δ_H (8)	θ (9)	Δ_V (10)	Δ_H (11)	θ (12)
(a) $r/t = 100$												
$r/l = 0.1$:												
30	12.53	0.1445	0.1108	−0.1445	−0.0585	−0.04556	0.3401	0.000480	0.000363	−1.330	−0.5469	−0.5144
35	6.265	0.2508	0.1420	−0.2508	−0.1231	−0.07108	0.2178	0.001006	0.000544	−1.704	−0.8531	−0.5964
40	3.611	0.4009	0.1752	−0.4009	−0.2325	−0.1039	0.1443	0.002009	0.000822	−2.102	−1.247	−0.6749
45	2.591	0.5987	0.2088	−0.5981	−0.4040	−0.1444	0.1084	0.003771	0.001223	−2.505	−1.734	−0.7542
50	2.043	0.8434	0.2410	−0.8434	−0.6558	−0.1925	0.0792	0.006516	0.001776	−2.892	−2.312	−0.8290
$r/l = 0.2$:												
30	17.38	2.031	0.0982	−2.031	−0.8742	−0.04288	0.6950	0.00726	0.000353	−18.86	−8.233	−0.4893
35	12.87	3.538	0.1259	−3.538	−1.794	−0.06530	0.4733	0.02065	0.000678	−24.18	−12.54	−0.5564
40	12.51	5.544	0.1524	−5.544	−3.282	−0.09256	0.3715	0.04590	0.001145	−29.25	−17.77	−0.6159
45	13.47	7.956	0.1748	−7.956	−5.447	−0.1233	0.3402	0.08862	0.001757	−33.56	−23.67	−0.6651
50	14.54	10.50	0.1899	−10.50	−8.270	−0.1546	0.3610	0.1526	0.002478	−36.46	−29.70	−0.6986
$r/l = 0.3$:												
30	35.24	9.474	0.0891	−9.474	−4.010	−0.03866	1.187	0.06357	0.000536	−86.64	−37.58	−0.4499
35	37.31	15.65	0.1092	−15.65	−7.890	−0.05606	0.9583	0.1474	0.000923	−106.2	−54.50	−0.4965
40	42.38	22.99	0.1245	−22.99	−13.59	−0.07595	0.9561	0.2931	0.001413	−121.0	−73.83	−0.5288
45	46.06	30.12	0.1317	−30.12	−20.69	−0.09384	1.073	0.5020	0.001927	−127.9	−91.21	−0.5434
50	45.75	35.11	0.1285	−35.11	−27.93	−0.1070	1.209	0.7422	0.002334	−124.9	−104.0	−0.5402
$r/l = 0.4$:												
30	75.06	26.10	0.07760	−26.10	−11.06	−0.03483	1.940	0.2256	0.000632	−239.2	−104.0	−0.4049
35	87.57	40.75	0.09046	−40.75	−20.25	−0.04716	1.924	0.5080	0.000994	−277.7	−145.2	−0.4320
40	96.68	55.00	0.09596	−55.00	−32.68	−0.05921	2.189	0.9067	0.001384	−293.8	−181.4	−0.4417
45	96.10	64.69	0.09256	−64.69	−44.95	−0.06746	2.500	1.372	0.001641	−284.1	−207.1	−0.4348
50	85.15	67.32	0.08360	−67.32	−54.74	−0.07216	2.591	1.731	0.001732	−255.7	−220.8	−0.4228

$r/l = 0.5$:												
30	142.2	54.14	0.06634	-54.14	-22.98	-0.02900	3.218	0.5995	0.000644	-497.6	-216.6	-0.3602
35	163.2	78.77	0.07239	-78.77	-39.97	-0.03811	3.588	1.202	0.000945	-542.9	-285.9	-0.3706
40	167.7	97.43	0.07124	-97.43	-58.47	-0.04484	4.141	1.938	0.001160	-535.2	-336.5	-0.3668
45	152.4	104.9	0.06505	-104.9	-74.31	-0.04883	4.418	2.566	0.001242	-487.9	-366.2	-0.3568
50	127.5	103.9	0.05747	-103.9	-87.21	-0.05166	4.246	2.945	0.001199	-431.0	-387.5	-0.3493
$r/l = 0.6$:												
30	238.5	88.47	0.05559	-88.47	-39.78	-0.02434	5.112	1.218	0.000620	-864.6	-378.6	-0.3247
35	257.7	126.7	0.05714	-126.7	-64.74	-0.03049	5.976	2.242	0.000834	-888.7	-474.2	-0.3185
40	245.4	144.9	0.05366	-144.9	-88.10	-0.03396	6.629	3.250	0.000932	-834.6	-528.2	-0.3171
45	212.3	149.5	0.04783	-149.5	-108.3	-0.03700	6.601	3.942	0.000918	-743.9	-575.4	-0.3042
50	175.8	143.7	0.04304	-147.9	-127.9	-0.03967	6.678	4.324	0.000865	-669.4	-618.7	-0.3086

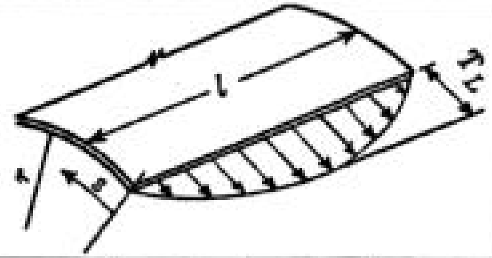
(b) $r/t = 200$

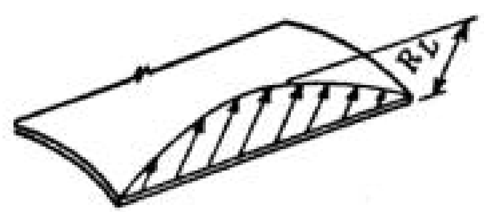
$r/l = 0.1$:												
30	13.59	0.5598	0.10563	-0.5598	-0.3112	-0.04555	0.3446	0.001423	0.000252	-5.070	-2.186	-0.5143
35	7.732	0.9838	0.13813	-0.9838	-0.4916	-0.07100	0.2228	0.003352	0.000438	-6.630	-3.408	-0.5958
40	5.711	1.576	0.1719	-1.576	-0.9269	-0.1036	0.1579	0.007139	0.000718	-8.250	-4.974	-0.6749
45	5.186	2.354	0.2050	-2.354	-1.462	-0.1434	0.1237	0.01391	0.001117	-9.842	-6.883	-0.7503
50	5.265	3.292	0.2352	-3.292	-2.576	-0.1895	0.1083	0.02506	0.001649	-11.29	-9.097	-0.8195
$r/l = 0.2$:												
30	32.02	8.267	0.0986	-8.267	-3.492	-0.04257	0.7716	0.03561	0.000385	-75.70	-32.70	-0.4866
35	33.96	14.07	0.1245	-14.07	-7.090	-0.06423	0.6124	0.08739	0.000700	-95.68	-49.33	-0.5497
40	39.51	21.38	0.1468	-21.38	-12.62	-0.08898	0.6071	0.1815	0.001123	-112.7	-68.34	-0.5990
45	44.27	28.96	0.1600	-28.96	-19.84	-0.1132	0.6944	0.3237	0.001601	-122.9	-86.91	-0.6273
50	45.16	34.63	0.1600	-34.63	-27.44	-0.1321	0.8010	0.4950	0.002014	-122.8	-101.0	-0.6299
$r/l = 0.3$:												
30	110.9	37.12	0.08756	-37.12	-15.70	-0.03796	1.641	0.2500	0.000532	-340.4	-147.6	-0.4443
35	123.0	58.70	0.10342	-58.70	-29.63	-0.05150	1.782	0.5520	0.000868	-402.2	-200.2	-0.4792
40	136.4	78.62	0.10899	-78.62	-46.73	-0.06707	2.154	0.9824	0.001193	-423.7	-260.8	-0.4884
45	131.1	89.18	0.10274	-89.18	-62.16	-0.07481	2.432	1.410	0.001376	-399.4	-290.9	-0.4749
50	111.5	89.27	0.09013	-89.27	-73.18	-0.07807	2.437	1.695	0.001382	-350.4	-303.5	-0.4550
$r/l = 0.4$:												
30	247.6	98.82	0.07423	-98.82	-41.89	-0.03237	3.525	0.9471	0.000586	-912.9	-397.8	-0.3936
35	283.3	139.7	0.07945	-139.7	-70.98	-0.04180	4.354	1.701	0.000837	-976.3	-513.6	-0.4011
40	273.8	161.7	0.07471	-161.7	-97.64	-0.04717	4.977	2.518	0.000964	-918.0	-579.7	-0.3884
45	232.0	163.1	0.06551	-163.1	-117.2	-0.04975	4.925	3.016	0.000945	-805.0	-611.3	-0.3732
50	187.9	157.7	0.05768	-157.7	-135.5	-0.05270	4.456	3.239	0.000875	-708.8	-647.5	-0.3670
$r/l = 0.5$:												
30	473.4	194.4	0.06058	-194.4	-82.77	-0.02666	6.899	2.116	0.000570	-1817.	-800.0	-0.3421
35	486.4	244.3	0.05905	-244.3	-125.5	-0.03167	8.309	3.527	0.000704	-1772.	-950.2	-0.3343
40	424.2	256.6	0.05240	-256.6	-158.6	-0.03414	8.519	4.483	0.000707	-1572.	-1024.	-0.3210
45	349.1	254.2	0.04653	-254.2	-189.1	-0.03671	7.792	4.924	0.000653	-1396.	-1101.	-0.3156
50	295.1	260.4	0.04398	-260.4	-234.5	-0.04173	6.701	5.075	0.000616	-1319.	-1252.	-0.3226
$r/l = 0.6$:												
30	756.0	315.3	0.04838	-315.3	-135.1	-0.02155	11.82	3.979	0.000510	-3009.	-1341.	-0.2961
35	706.1	360.6	0.04430	-360.6	-188.1	-0.02429	13.15	5.790	0.000555	-2756.	-1511.	-0.2847
40	591.7	367.3	0.03930	-367.3	-232.8	-0.02637	12.46	6.714	0.000519	-2445.	-1640.	-0.2778
45	498.8	374.8	0.03615	-374.8	-287.2	-0.02931	11.18	7.265	0.000486	-2249.	-1823	-0.2782
50	431.9	389.3	0.03392	-389.3	-357.3	-0.03284	10.21	7.983	0.000481	-2110.	-2043.	-0.2807

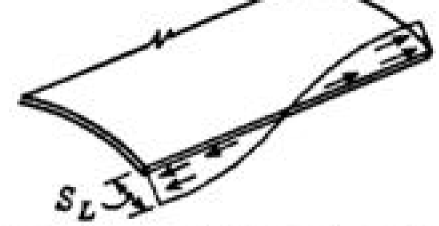
TABLE 3A.—EDGE LOADS ON SIMPLY SUPPORTED CYLINDRICAL SHELLS ($n = 1$)

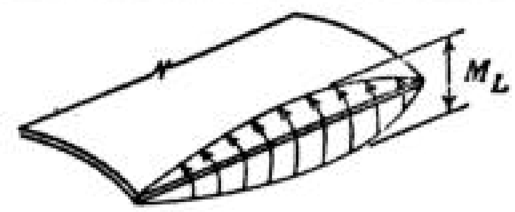
TANGENTIAL EDGE LOAD	RADIAL EDGE LOAD	SHEAR EDGE LOAD	MOMENT EDGE LOAD
(a) BASIC FORMULAS FOR $n = 1$; FOR $n = 3$ SUBSTITUTE $\frac{3\pi x}{l}$ INSTEAD OF $\frac{\pi x}{l}$			
Longitudinal Force T_x— $T_L \times$ Col. (1) $\times \sin \frac{\pi x}{l}$	Longitudinal Force T_x— $R_L \times \frac{l}{t} \times$ Col. (5) $\times \sin \frac{\pi x}{l}$	Longitudinal Force T_x— $S_L \times$ Col. (9) $\times \sin \frac{\pi x}{l}$	Longitudinal Force T_x— $M_L \times \frac{1}{t} \times$ Col. (13) $\times \sin \frac{\pi x}{l}$
Shearing Force S— $T_L \times$ Col. (2) $\times \cos \frac{\pi x}{l}$	Shearing Force S— $R_L \times \frac{l}{t} \times$ Col. (6) $\times \cos \frac{\pi x}{l}$	Shearing Force S— $S_L \times$ Col. (10) $\times \cos \frac{\pi x}{l}$	Shearing Force S— $M_L \times \frac{1}{t} \times$ Col. (14) $\times \cos \frac{\pi x}{l}$
Transverse Force $T\phi$— $T_L \times$ Col. (3) $\times \sin \frac{\pi x}{l}$	Transverse Force $T\phi$— $R_L \times \frac{l}{t} \times$ Col. (7) $\times \sin \frac{\pi x}{l}$	Transverse Force $T\phi$— $S_L \times$ Col. (11) $\times \sin \frac{\pi x}{l}$	Transverse Force $T\phi$— $M_L \times \frac{1}{t} \times$ Col. (15) $\times \sin \frac{\pi x}{l}$
Transverse Moment $M\phi$— $T_L \times t \times$ Col. (4) $\times \sin \frac{\pi x}{l}$	Transverse Moment $M\phi$— $R_L \times l \times$ Col. (8) $\times \sin \frac{\pi x}{l}$	Transverse Moment $M\phi$— $S_L \times t \times$ Col. (12) $\times \sin \frac{\pi x}{l}$	Transverse Moment $M\phi$— $M_L \times$ Col. (16) $\times \sin \frac{\pi x}{l}$

(b) $n = 1$









$\frac{s}{\sqrt[4]{r t l^2}}$	T_x (1)	S (2)	$T\phi$ (3)	$M\phi$ (4)	T_x (5)	S (6)	$T\phi$ (7)	$M\phi$ (8)	T_x (9)	S (10)	$T\phi$ (11)	$M\phi$ (12)	T_x (13)	S (14)	$T\phi$ (15)	$M\phi$ (16)
$\frac{rt}{l^2} = 0.002$:																
3.2	− 0.595	−0.337	+0.032	+0.3010	+0.0209	+0.00594	+0.00026	−0.00393	−0.012	−0.039	+0.0084	+0.0419	+0.1022	+0.0127	+0.00503	−0.001
1.6	+ 0.635	+1.303	−0.188	−1.8271	−0.0633	−0.02668	−0.00011	+0.03136	−0.204	+0.123	−0.0368	−0.2418	−0.3372	−0.0526	−0.02517	+0.075
0.8	−10.323	−1.441	−0.434	−2.1133	+0.1676	−0.00575	+0.01432	+0.07662	−0.788	−0.260	−0.0180	−0.1408	+0.4747	−0.0790	+0.02973	+0.534
0.4	− 2.658	−3.637	+0.283	−0.5169	+0.1185	+0.04052	+0.00946	+0.05128	+1.580	−0.230	−0.0607	+0.0162	+0.6238	+0.0956	+0.02863	+0.761
0.2	+11.837	−3.132	+0.754	+0.0620	−0.1123	+0.04313	+0.00356	+0.02703	+4.371	+0.150	+0.0701	+0.0348	−0.1552	+0.1370	+0.01205	+0.872
0.1	+23.039	−1.990	+0.929	+0.1309	−0.3127	+0.02938	+0.00108	+0.01362	+6.321	+0.503	+0.0491	+0.0222	−0.9661	+0.1017	+0.00382	+0.932
0.0	+37.430	0	+1.000	0	−0.5848	0	0	0	+8.706	+1.000	0	0	−2.1086	0	0	+1.000

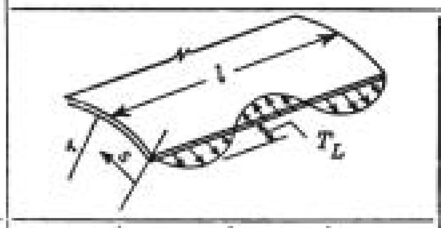
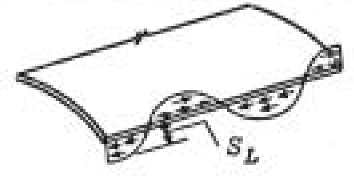
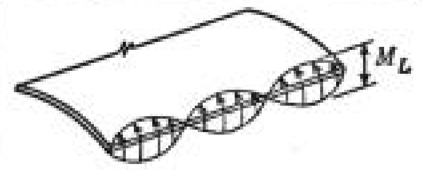
$\frac{rt}{l^2} = 0.003$:																
3.2	− 0.393	−0.288	+0.038	+0.2387	+0.0203	+0.00695	+0.00015	−0.00432	+0.000	−0.036	+0.0098	+0.0360	+0.0904	+0.0132	+0.00529	−0.002
1.6	+ 0.316	+1.115	−0.204	−1.4220	−0.0629	−0.03104	+0.00041	+0.03275	−0.205	+0.113	−0.0421	−0.2075	−0.3010	−0.0517	−0.02790	+0.066
0.8	− 8.063	−1.331	−0.405	−1.5481	+0.1741	−0.00573	+0.01853	+0.07812	−0.665	−0.261	−0.0160	−0.1067	+0.4217	−0.0796	+0.03239	+0.493
0.4	− 1.920	−3.205	+0.305	−0.3150	+0.1214	+0.04716	+0.01212	+0.05225	+1.451	−0.219	+0.0697	+0.0260	+0.5681	+0.0942	+0.03157	+0.728
0.2	+ 9.454	−2.742	+0.762	+0.1030	−0.1185	+0.04990	+0.00455	+0.02758	+3.919	+0.161	+0.0785	+0.0369	−0.1315	+0.1367	+0.01334	+0.852
0.1	+18.230	−1.739	+0.931	+0.1323	−0.3269	+0.03395	+0.00138	+0.01394	+5.642	+0.510	+0.0546	+0.0227	−0.8691	+0.1019	+0.00424	+0.922
0.0	+29.509	0	+1.000	0	−0.6104	0	0	0	+7.750	+1.000	0	0	−1.9706	0	0	+1.000
$\frac{rt}{l^2} = 0.004$:																
3.2	− 0.278	−0.254	+0.042	+0.2008	+0.0196	+0.00771	+0.00000	−0.00459	+0.008	−0.033	+0.0108	+0.0320	+0.0817	+0.0134	+0.00539	−0.002
1.6	+ 0.142	+0.991	−0.215	−1.1832	−0.0618	−0.03434	+0.00103	+0.03355	−0.205	+0.106	−0.0461	−0.1849	−0.2743	−0.0502	−0.02974	+0.060
0.8	− 6.733	−1.259	−0.381	−1.2254	+0.1776	−0.00550	+0.02214	+0.07856	−0.585	−0.262	−0.0136	−0.0851	+0.3819	−0.0794	+0.03386	+0.462
0.4	− 1.500	−2.923	+0.321	−0.2052	+0.1226	+0.05223	+0.01437	+0.05255	+1.366	−0.208	+0.0771	+0.0318	+0.5264	+0.0918	+0.03339	+0.702
0.2	+ 8.039	−2.488	+0.769	+0.1217	−0.1224	+0.05501	+0.00539	+0.02779	+3.624	+0.169	+0.0851	+0.0378	−0.1135	+0.1348	+0.01418	+0.837
0.1	+15.393	−1.575	+0.933	+0.1301	−0.3353	+0.03740	+0.00163	+0.01407	+5.200	+0.515	+0.0589	+0.0227	−0.7959	+0.1008	+0.00452	+0.913
0.0	+24.846	0	+1.000	0	−0.6254	0	0	0	+7.129	+1.000	0	0	−1.8212	0	0	+1.000
$\frac{rt}{l^2} = 0.006$:																
3.2	− 0.153	−0.209	+0.047	+0.1550	+0.0179	+0.00877	−0.00034	−0.00493	+0.017	−0.029	+0.0122	+0.0265	+0.0692	+0.0134	+0.00536	−0.002
1.6	− 0.103	+0.827	−0.230	−0.9042	−0.0591	−0.03918	+0.00239	+0.03457	−0.202	+0.094	−0.0522	−0.1554	−0.2359	−0.0467	−0.03216	+0.050
0.8	− 5.182	−1.161	−0.344	−0.8634	+0.1807	−0.00480	+0.02822	+0.07956	−0.481	−0.263	−0.0089	−0.0585	+0.3234	−0.0780	+0.03503	+0.417
0.4	− 1.030	−2.556	+0.347	−0.0900	+0.1227	+0.05977	+0.01810	+0.05372	+1.253	−0.197	+0.0890	+0.0380	+0.4651	+0.0860	+0.03538	+0.664
0.2	+ 6.371	−2.158	+0.778	+0.1355	−0.1270	+0.06253	+0.00677	+0.02870	+3.240	+0.182	+0.0955	+0.0382	−0.0872	+0.1294	+0.01514	+0.814
0.1	+12.071	−1.363	+0.936	+0.1232	−0.3444	+0.04246	+0.00205	+0.01463	+4.628	+0.523	+0.0655	+0.0224	−0.6882	+0.0975	+0.00484	+0.901
0.0	+19.407	0	+1.000	0	−0.6412	0	0	0	+6.328	+1.000	0	0	−1.6010	0	0	+1.000
$\frac{rt}{l^2} = 0.008$:																
3.2	− 0.086	−0.178	+0.050	+0.1271	+0.0163	+0.00949	−0.00072	−0.00511	+0.022	−0.025	+0.0132	+0.0227	+0.0605	+0.0132	+0.00522	−0.002
1.6	− 0.131	+0.719	−0.240	−0.7409	−0.0562	−0.04265	+0.00383	+0.03459	−0.199	+0.084	−0.0567	−0.1360	−0.2087	−0.0432	−0.03372	+0.043
0.8	− 4.278	−1.094	−0.313	−0.6617	+0.1813	−0.00397	+0.03329	+0.07719	−0.414	−0.263	−0.0042	−0.0422	+0.2814	−0.0763	+0.03497	+0.385
0.4	− 0.770	−2.316	+0.367	−0.0315	+0.1217	+0.06531	+0.02116	+0.05188	+1.177	−0.187	+0.0988	+0.0410	+0.4211	+0.0803	+0.03618	+0.637
0.2	+ 5.382	−1.945	+0.785	+0.1383	−0.1293	+0.06797	+0.00791	+0.02768	+2.990	+0.191	+0.1036	+0.0379	−0.0676	+0.1235	+0.01561	+0.796
0.1	+10.120	−1.226	+0.938	+0.1163	−0.3483	+0.04611	+0.00239	+0.01411	+4.256	+0.530	+0.0707	+0.0218	−0.6093	+0.0937	+0.00501	+0.892
0.0	+16.225	0	+1.000	0	−0.6479	0	0	0	+5.810	+1.000	0	0	−1.4397	0	0	+1.000
$\frac{rt}{l^2} = 0.010$:																
3.2	− 0.045	−0.155	+0.052	+0.1079	+0.0148	+0.00997	−0.00111	−0.00521	+0.026	−0.022	+0.0138	+0.0199	+0.0539	+0.0128	+0.00503	−0.002
1.6	− 0.183	+0.640	−0.246	−0.6312	−0.0534	−0.04527	+0.00527	+0.03458	−0.196	+0.076	−0.0601	−0.1218	−0.1880	−0.0399	−0.03485	+0.038
0.8	− 3.671	−1.043	−0.288	−0.5322	+0.1807	−0.00313	+0.03767	+0.07597	−0.365	−0.262	+0.0004	−0.0312	+0.2487	−0.0746	+0.03426	+0.359
0.4	− 0.604	−2.141	+0.384	+0.0026	+0.1202	+0.06960	+0.02378	+0.05126	+1.121	−0.178	+0.1071	+0.0425	+0.3866	+0.0748	+0.03637	+0.614
0.2	+ 4.712	−1.789	+0.791	+0.1370	−0.1305	+0.07215	+0.00887	+0.02748	+2.807	+0.199	+0.1105	+0.0372	−0.0524	+0.1178	+0.01581	+0.783
0.1	+ 8.807	−1.127	+0.940	+0.1100	−0.3496	+0.04892	+0.00268	+0.01406	+3.986	+0.534	+0.0750	+0.0212	−0.5477	+0.0899	+0.00509	+0.884
0.0	+14.089	0	+1.000	0	−0.6501	0	0	0	+5.435	+1.000	0	0	−1.3134	0	0	+1.000

TABLE 3A.—(Continued)

$\frac{s}{\sqrt[4]{rtl^2}}$	Tangential Edge Load				Radial Edge Load				Shear Edge Load				Moment Edge Load			
	(b) (Continued)															
	T_x (1)	S (2)	$T\phi$ (3)	$M\phi$ (4)	T_x (5)	S (6)	$T\phi$ (7)	$M\phi$ (8)	T_x (9)	S (10)	$T\phi$ (11)	$M\phi$ (12)	T_x (13)	S (14)	$T\phi$ (15)	$M\phi$ (16)
$\frac{rt}{l^2} = 0.015$:																
3.2	+0.011	−0.115	+0.055	+0.0775	+0.0117	+0.01060	−0.00205	−0.00527	+0.030	−0.016	+0.0146	+0.0149	+0.0425	+0.0119	+0.00455	−0.002
1.6	−0.243	+0.504	−0.255	−0.4648	−0.0474	−0.04966	+0.00882	+0.03405	−0.187	+0.059	−0.0657	−0.0980	−0.1518	−0.0325	−0.03674	+0.028
0.8	−2.753	−0.952	−0.236	−0.3476	+0.1769	−0.00114	+0.04659	+0.07280	−0.284	−0.260	+0.0112	−0.0145	+0.1904	−0.0708	+0.03121	+0.313
0.4	−0.371	−1.845	+0.416	+0.0440	+0.1159	+0.07720	+0.02902	+0.04970	+1.023	−0.160	+0.1242	+0.0435	+0.3249	+0.0626	+0.03557	+0.573
0.2	+3.678	−1.529	+0.803	+0.1287	−0.1309	+0.07946	+0.01080	+0.02699	+2.498	+0.214	+0.1242	+0.0352	−0.0252	+0.1047	+0.01574	+0.757
0.1	+6.801	−0.961	+0.943	+0.0969	−0.3472	+0.05383	+0.00326	+0.01393	+3.533	+0.544	+0.0836	+0.0197	−0.4365	+0.0811	+0.00511	+0.869
0.0	+10.843	0	+1.000	0	−0.6465	0	0	0	+4.811	+1.000	0	0	−1.0849	0	0	+1.000
$\frac{rt}{l^2} = 0.020$:																
3.2	+0.033	−0.088	+0.054	+0.0594	+0.0091	+0.01073	−0.00290	−0.00518	+0.031	−0.011	+0.0147	+0.0116	+0.0352	+0.0111	+0.00412	−0.002
1.6	−0.263	+0.415	−0.258	−0.3689	−0.0424	−0.05224	+0.01217	+0.03322	−0.178	+0.047	−0.0689	−0.0825	−0.1279	−0.0264	−0.03805	+0.022
0.8	−2.225	−0.888	−0.195	−0.2500	+0.1720	+0.00061	+0.05358	+0.06986	−0.233	−0.257	+0.0212	−0.0052	+0.1510	−0.0676	+0.02741	+0.281
0.4	−0.250	−1.652	+0.440	+0.0601	+0.1117	+0.08214	+0.03306	+0.04833	+0.957	−0.146	+0.1381	+0.0428	+0.2824	+0.0523	+0.03381	+0.544
0.2	+3.068	−1.361	+0.812	+0.1192	−0.1296	+0.08417	+0.01230	+0.02659	+2.297	+0.225	+0.1350	+0.0331	−0.0068	+0.0934	+0.01527	+0.738
0.1	+5.634	−0.854	+0.946	+0.0868	−0.3426	+0.05700	+0.00371	+0.01384	+3.241	+0.550	+0.0902	+0.0183	−0.3599	+0.0735	+0.00500	+0.859
0.0	+8.964	0	+1.000	0	−0.6374	0	0	0	+4.410	+1.000	0	0	−0.9265	0	0	+1.000
$\frac{rt}{l^2} = 0.025$:																
3.2	+0.044	−0.069	+0.053	+0.0472	+0.0071	+0.01060	−0.00364	−0.00502	+0.031	−0.008	+0.0143	+0.0092	+0.0301	+0.0103	+0.00376	−0.001
1.6	−0.268	+0.351	−0.257	−0.3054	−0.0382	−0.05380	+0.01529	+0.03229	−0.170	+0.036	−0.0705	−0.0715	−0.1107	−0.0213	−0.03912	+0.017
0.8	−1.875	−0.839	−0.161	−0.1901	+0.1667	+0.00212	+0.05928	+0.06722	−0.197	−0.253	+0.0305	+0.0005	+0.1221	−0.0649	+0.02345	+0.257
0.4	−0.177	−1.512	+0.460	+0.0668	+0.1078	+0.08554	+0.03631	+0.04747	+0.906	−0.135	+0.1499	+0.0416	+0.2507	+0.0436	+0.03181	+0.521
0.2	+2.656	−1.240	+0.819	+0.1104	−0.1275	+0.08738	+0.01350	+0.02630	+2.150	+0.234	+0.1440	+0.0312	+0.0067	+0.0838	+0.01466	+0.723
0.1	+4.853	−0.777	+0.948	+0.0787	−0.3362	+0.05918	+0.00407	+0.01380	+3.029	+0.556	+0.0958	+0.0170	−0.3028	+0.0670	+0.00484	+0.850
0.0	+7.714	0	+1.000	0	−0.6263	0	0	0	+4.123	+1.000	0	0	−0.8076	0	0	+1.000
$\frac{rt}{l^2} = 0.030$:																
3.2	+0.050	−0.054	+0.051	+0.0383	+0.0054	+0.01031	−0.00428	−0.00482	+0.030	−0.005	+0.0137	+0.0073	+0.0263	+0.0096	+0.00345	−0.001
1.6	−0.265	+0.301	−0.254	−0.2599	−0.0346	−0.05470	+0.01818	+0.03133	−0.163	+0.028	−0.0712	−0.0630	−0.0976	−0.0170	−0.04002	+0.013
0.8	−1.625	−0.798	−0.131	−0.1497	+0.1616	+0.00344	+0.06403	+0.06485	−0.169	−0.250	+0.0392	+0.0044	+0.0998	−0.0627	+0.01962	+0.237
0.4	−0.129	−1.403	+0.477	+0.0693	+0.1043	+0.08796	+0.03900	+0.04617	+0.866	−0.125	+0.1604	+0.0402	+0.2259	+0.0360	+0.02977	+0.502
0.2	+2.355	−1.147	+0.825	+0.1028	−0.1249	+0.08966	+0.01449	+0.02607	+2.035	+0.242	+0.1518	+0.0294	+0.0171	+0.0754	+0.01401	+0.710
0.1	+4.287	−0.719	+0.949	+0.0722	−0.3292	+0.06074	+0.00437	+0.01379	+2.866	+0.560	+0.1005	+0.0159	−0.2581	+0.0613	+0.00467	+0.843
0.0	+6.811	0	+1.000	0	−0.6144	0	0	0	+3.902	+1.000	0	0	−0.7141	0	0	+1.000

$\frac{rt}{l^2} = 0.040$:																
3.2	+ 0.053	−0.033	+0.045	+0.0263	+0.0028	+0.00947	−0.00526	−0.00439	+0.028	+0.000	+0.0120	+0.0047	+0.0210	+0.0085	+0.00297	−0.000
1.6	− 0.253	+0.227	−0.246	−0.1986	−0.0289	−0.05524	+0.02329	+0.02949	−0.150	+0.014	−0.0704	−0.0507	−0.0787	−0.0100	−0.04169	+0.008
0.8	− 1.286	−0.735	−0.081	−0.1002	+0.1516	+0.00551	+0.07145	+0.06091	−0.131	−0.242	+0.0550	+0.0088	+0.0678	−0.0589	+0.01218	+0.208
0.4	− 0.072	−1.241	+0.505	+0.0682	+0.0980	+0.09076	+0.04319	+0.04469	+0.804	−0.109	+0.1781	+0.0372	+0.1887	+0.0238	+0.02556	+0.473
0.2	+ 1.938	−1.010	+0.834	+0.0895	−0.1195	+0.09230	+0.01605	+0.02586	+1.866	+0.253	+0.1648	+0.0263	+0.0315	+0.0615	+0.01258	+0.690
0.1	+ 3.512	−0.632	+0.952	+0.0618	−0.3152	+0.06258	+0.00485	+0.01389	+2.625	+0.567	+0.1085	+0.0140	−0.1925	+0.0517	+0.00426	+0.831
0.0	+ 5.579	0	+1.000	0	−0.5904	0	0	0	+3.580	+1.000	0	0	−0.5749	0	0	+1.000
$\frac{rt}{l^2} = 0.060$:																
3.2	+ 0.047	−0.009	+0.034	+0.0135	−0.0003	+0.00747	−0.00631	−0.00356	+0.023	+0.005	+0.0081	+0.0017	+0.0151	+0.0070	+0.00225	−0.001
1.6	− 0.222	+0.137	−0.222	−0.1313	−0.0213	−0.05387	+0.03126	+0.02620	−0.130	−0.005	−0.0646	−0.0359	−0.0563	−0.0007	−0.04437	+0.001
0.8	− 0.910	−0.645	−0.005	−0.0531	+0.1343	+0.00814	+0.08098	+0.05495	−0.085	−0.230	+0.0822	+0.0122	+0.0302	−0.0533	−0.00023	+0.170
0.4	− 0.023	−1.035	+0.545	+0.0605	+0.0881	+0.09213	+0.04853	+0.04274	+0.718	−0.086	+0.2059	+0.0319	+0.1416	+0.0065	+0.01821	+0.431
0.2	+ 1.456	−0.838	+0.848	+0.0708	−0.1084	+0.09368	+0.01807	+0.02586	+1.647	+0.269	+0.1850	+0.0216	+0.0474	+0.0415	+0.01001	+0.661
0.1	+ 2.629	−0.524	+0.956	+0.0480	−0.2886	+0.06370	+0.00547	+0.01425	+2.321	+0.575	+0.1207	+0.0112	−0.1122	+0.0378	+0.00352	+0.814
0.0	+ 4.187	0	+1.000	0	−0.5452	0	0	0	+3.179	+1.000	0	0	−0.4003	0	0	+1.000
$\frac{rt}{l^2} = 0.080$:																
3.2	+ 0.039	+0.002	+0.023	+0.0072	−0.0018	+0.00559	−0.00651	−0.00285	+0.018	+0.008	+0.0044	+0.0002	+0.0118	+0.0061	+0.00166	−0.000
1.6	− 0.195	+0.083	−0.196	−0.0951	−0.0166	−0.05117	+0.03683	+0.02346	−0.114	−0.018	−0.0564	−0.0271	−0.0433	+0.0050	−0.04634	−0.002
0.8	− 0.705	−0.581	+0.052	−0.0320	+0.1199	+0.00946	+0.08620	+0.05066	−0.060	−0.218	+0.1045	+0.0128	+0.0100	−0.0491	−0.00982	+0.146
0.4	− 0.005	−0.904	+0.574	+0.0523	+0.0801	+0.09073	+0.05150	+0.04162	+0.659	−0.071	+0.2273	+0.0277	+0.1128	−0.0047	+0.01226	+0.402
0.2	+ 1.179	−0.730	+0.857	+0.0582	−0.0980	+0.09256	+0.01923	+0.02615	+1.506	+0.278	+0.2004	+0.0182	+0.0549	+0.0280	+0.00787	+0.639
0.1	+ 2.130	−0.458	+0.959	+0.0391	−0.2648	+0.06315	+0.00583	+0.01471	+2.130	+0.580	+0.1301	+0.0093	−0.0654	+0.0283	+0.00289	+0.801
0.0	+ 3.407	0	+1.000	0	−0.5051	0	0	0	+2.931	+1.000	0	0	−0.2949	0	0	+1.000
$\frac{rt}{l^2} = 0.100$:																
3.2	+ 0.032	+0.008	+0.015	+0.0038	−0.0026	+0.00400	−0.00620	−0.00229	+0.015	+0.009	+0.0014	−0.0006	+0.0097	+0.0056	+0.00110	−0.000
1.6	− 0.171	+0.048	−0.171	−0.0728	−0.0135	−0.04807	+0.04061	+0.02117	−0.102	−0.026	−0.0474	−0.0214	−0.0347	+0.0086	−0.04764	−0.004
0.8	− 0.576	−0.533	+0.097	−0.0210	+0.1078	+0.00995	+0.08883	+0.04740	−0.045	−0.209	+0.1231	+0.0126	−0.0027	−0.0456	−0.01715	+0.128
0.4	+ 0.002	−0.811	+0.595	+0.0454	+0.0747	+0.08805	+0.05308	+0.04093	+0.614	−0.059	+0.2446	+0.0243	+0.0922	−0.0123	+0.00751	+0.380
0.2	+ 0.997	−0.655	+0.865	+0.0490	−0.0886	+0.09028	+0.01989	+0.02656	+1.406	+0.285	+0.2129	+0.0156	+0.0578	+0.0183	+0.00611	+0.622
0.1	+ 1.807	−0.411	+0.961	+0.0328	−0.2437	+0.06183	+0.00605	+0.01520	+1.995	+0.583	+0.1378	+0.0078	−0.0360	+0.0213	+0.00237	+0.790
0.0	+ 2.903	0	+1.000	0	−0.4697	0	0	0	+2.760	+1.000	0	0	−0.2250	0	0	+1.000

TABLE 3A.—(Continued) (For $n = 3$)

$\frac{s}{\sqrt[4]{r\,t\,l^2}}$	Tangential Edge Load				Radial Edge Load				Shear Edge Load				Moment Edge Load			
	(c) $n = 3$															
																
Ratio	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
$\frac{r\,t}{l^2} = 0.002$:																
1.6	+ 0.285	−0.025	+0.090	+0.0576	−0.0034	+0.00384	−0.00282	−0.00287	+0.082	+0.015	+0.0153	+0.0020	+0.0364	+0.0312	−0.00623	−0.014
0.8	− 0.902	+0.307	−0.353	−0.4678	−0.0010	−0.01922	+0.00817	+0.01637	−0.309	−0.009	−0.0737	−0.0861	−0.1255	−0.0620	−0.02633	+0.069
0.4	− 2.287	−1.204	−0.080	−0.1920	+0.0635	+0.00753	+0.01653	+0.02281	−0.069	−0.278	+0.0506	+0.0107	+0.2392	−0.0438	+0.03600	+0.352
0.2	+ 0.448	−1.717	+0.538	+0.0868	+0.0227	+0.02868	+0.00880	+0.01449	+1.288	−0.082	+0.1401	+0.0436	+0.2567	+0.0734	+0.03054	+0.601
0.1	+ 3.947	−1.310	+0.851	+0.1194	−0.0602	+0.02591	+0.00309	+0.00779	+2.610	+0.298	+0.1229	+0.0304	−0.0956	+0.0961	+0.01249	+0.775
0.0	+ 9.616	0	+1.000	0	−0.2138	0	0	0	+4.553	+1.000	0	0	−0.9839	0	0	+1.000
$\frac{r\,t}{l^2} = 0.003$:																
1.6	+ 0.221	+0.009	+0.071	+0.0296	−0.0042	+0.00318	−0.00328	−0.00243	+0.066	+0.020	+0.0111	−0.0019	+0.0287	+0.0267	−0.00637	−0.011
0.8	− 0.730	+0.195	−0.331	−0.3235	+0.0006	−0.01981	+0.01074	+0.01530	−0.267	−0.027	−0.0723	−0.0638	−0.1024	−0.0486	−0.03020	+0.051
0.4	− 1.664	−1.058	−0.020	−0.1060	+0.0600	+0.00878	+0.01978	+0.02138	−0.024	−0.267	+0.0697	+0.0174	+0.1734	−0.0444	+0.02937	+0.306
0.2	+ 0.399	−1.458	+0.567	+0.0889	+0.0211	+0.03082	+0.01044	+0.01395	+1.157	−0.061	+0.1609	+0.0406	+0.2128	+0.0560	+0.02767	+0.562
0.1	+ 3.028	−1.106	+0.861	+0.1027	−0.0583	+0.02775	+0.00366	+0.00767	+2.312	+0.313	+0.1378	+0.0272	−0.0527	+0.0796	+0.01165	+0.750
0.0	+ 7.321	0	+1.000	0	−0.2072	0	0	0	+4.028	+1.000	0	0	−0.7678	0	0	+1.000
$\frac{r\,t}{l^2} = 0.004$:																
1.6	+ 0.177	+0.026	+0.055	+0.0155	−0.0046	+0.00252	−0.00351	−0.00206	+0.054	+0.023	+0.0071	−0.0039	+0.0242	+0.0236	−0.00633	−0.008
0.8	− 0.619	+0.126	−0.308	−0.2442	+0.0016	−0.01982	+0.01281	+0.01433	−0.237	−0.039	−0.0686	−0.0503	−0.0878	−0.0386	−0.03359	+0.039
0.4	− 1.314	−0.959	+0.026	−0.0647	+0.0566	+0.00958	+0.02211	+0.02024	+0.002	−0.256	+0.0860	+0.0199	+0.1311	−0.0449	+0.02283	+0.274
0.2	+ 0.358	−1.290	+0.590	+0.0843	+0.0198	+0.03193	+0.01162	+0.01357	+1.069	−0.046	+0.1774	+0.0376	+0.1835	+0.0425	+0.02458	+0.534
0.1	+ 2.492	−0.975	+0.868	+0.0809	−0.0561	+0.02872	+0.00408	+0.00762	+2.119	+0.323	+0.1494	+0.0244	−0.0253	+0.0667	+0.01068	+0.732
0.0	+ 6.005	0	+1.000	0	−0.2000	0	0	0	+3.694	+1.000	0	0	−0.6246	0	0	+1.000
$\frac{r\,t}{l^2} = 0.006$:																
1.6	+ 0.122	+0.041	+0.031	+0.0025	−0.0047	+0.00138	−0.00353	−0.00150	+0.038	+0.025	+0.0003	−0.0055	+0.0190	+0.0195	−0.00617	−0.006
0.8	− 0.478	+0.045	−0.265	−0.1588	+0.0026	−0.01913	+0.01586	+0.01271	−0.196	−0.055	−0.0583	−0.0344	−0.0695	−0.0247	−0.03921	+0.025
0.4	− 0.929	−0.825	+0.096	−0.0271	+0.0507	+0.01044	+0.02518	+0.01853	+0.030	−0.239	+0.1132	+0.0210	+0.0798	−0.0452	+0.01138	+0.232
0.2	+ 0.296	−1.076	+0.622	+0.0726	+0.0180	+0.03269	+0.01317	+0.01309	+0.950	−0.024	+0.2032	+0.0322	+0.1453	+0.0231	+0.01888	+0.495
0.1	+ 1.875	−0.810	+0.879	+0.0716	−0.0515	+0.02942	+0.00463	+0.00763	+1.871	+0.337	+0.1674	+0.0202	+0.0074	+0.0477	+0.00882	+0.706
0.0	+ 4.514	0	+1.00	0	−0.1861	0	0	0	+3.277	+1.000	0	0	−0.4429	0	0	+1.000

$\frac{rt}{l^2} = 0.008$:																
1.6	+ 0.088	+0.044	+0.014	−0.0028	−0.0045	+0.00050	−0.00324	−0.00111	+0.027	+0.024	−0.0048	−0.0059	+0.0160	+0.0170	−0.00614	−0.005
0.8	− 0.390	−0.001	−0.226	−0.1141	+0.0030	−0.01811	+0.01791	+0.01142	−0.168	−0.064	−0.0468	−0.0253	−0.0581	−0.0158	−0.04352	+0.016
0.4	− 0.719	−0.736	+0.148	−0.0116	+0.0458	+0.01070	+0.02698	+0.01731	+0.043	−0.226	+0.1354	+0.0203	+0.0499	−0.0450	+0.00214	+0.205
0.2	+ 0.252	−0.940	+0.646	+0.0623	+0.0167	+0.03247	+0.01409	+0.01284	+0.871	−0.009	+0.2231	+0.0280	+0.1208	+0.0099	+0.01408	+0.467
0.1	+ 1.520	−0.707	+0.886	+0.0591	−0.0473	+0.02930	+0.00496	+0.00773	+1.713	+0.346	+0.1811	+0.0171	+0.0253	+0.0345	+0.00720	+0.686
0.0	+ 3.674	0	+1.000	0	−0.1735	0	0	0	+3.018	+1.000	0	0	−0.3316	0	0	+1.000
$\frac{rt}{l^2} = 0.010$:																
1.6	+ 0.066	+0.044	+0.002	−0.0050	−0.0042	−0.00015	−0.00282	−0.00083	+0.020	+0.023	−0.0084	−0.0058	+0.0139	+0.0152	−0.00623	−0.004
0.8	− 0.329	−0.030	−0.191	−0.0869	+0.0030	−0.01702	+0.01925	+0.01037	−0.147	−0.070	−0.0353	−0.0196	−0.0500	−0.0096	−0.04672	+0.011
0.4	− 0.588	−0.669	+0.189	−0.0042	+0.0416	+0.01064	+0.02798	+0.01639	+0.050	−0.214	+0.1539	+0.0190	+0.0308	−0.0443	−0.00521	+0.184
0.2	+ 0.218	−0.844	+0.663	+0.0539	+0.0157	+0.03177	+0.01463	+0.01271	+0.812	+0.002	+0.2394	+0.0247	+0.1032	+0.0006	+0.01013	+0.445
0.1	+ 1.288	−0.635	+0.892	+0.0501	−0.0434	+0.02880	+0.00516	+0.00786	+1.599	+0.352	+0.1924	+0.0147	+0.0359	+0.0249	+0.00584	+0.671
0.0	+ 3.131	0	+1.000	0	−0.1623	0	0	0	+2.838	+1.000	0	0	−0.2567	0	0	+1.000
$\frac{rt}{l^2} = 0.015$:																
1.6	+ 0.034	+0.035	−0.014	−0.0060	−0.0034	−0.00107	−0.00166	−0.00040	+0.009	+0.017	−0.0127	−0.0049	+0.0104	+0.0124	−0.00668	−0.003
0.8	− 0.236	−0.067	−0.121	−0.0512	+0.0026	−0.01450	+0.02065	+0.00844	−0.114	−0.076	−0.0097	−0.0118	−0.0366	−0.0007	−0.05099	+0.003
0.4	− 0.409	−0.558	+0.261	+0.0022	+0.0337	+0.00979	+0.02859	+0.01481	+0.055	−0.193	+0.1888	+0.0157	+0.0054	−0.0415	−0.01754	+0.150
0.2	+ 0.160	−0.692	+0.694	+0.0393	+0.0141	+0.02924	+0.01506	+0.01262	+0.710	+0.019	+0.2698	+0.0187	+0.0749	−0.0130	+0.00308	+0.406
0.1	+ 0.948	−0.523	+0.901	+0.0358	−0.0354	+0.02686	+0.00536	+0.00824	+1.415	+0.359	+0.2138	+0.0107	+0.0469	+0.0098	+0.00336	+0.642
0.0	+ 2.351	0	+1.000	0	−0.1390	0	0	0	+2.560	+1.000	0	0	−0.1481	0	0	+1.000
$\frac{rt}{l^2} = 0.020$:																
1.6	+ 0.019	+0.026	−0.018	−0.0053	−0.0027	−0.00140	−0.00071	−0.00019	+0.004	+0.012	−0.0131	−0.0039	+0.0081	+0.0104	−0.00702	−0.002
0.8	− 0.183	−0.081	−0.070	−0.0346	+0.0019	−0.01244	+0.02045	+0.00713	−0.094	−0.077	+0.0103	−0.0081	−0.0282	+0.0036	−0.05168	−0.001
0.4	− 0.320	−0.489	+0.306	+0.0033	+0.0280	+0.00858	+0.02783	+0.01380	+0.051	−0.179	+0.2123	+0.0130	−0.0060	−0.0379	−0.02428	+0.127
0.2	+ 0.123	−0.604	+0.711	+0.0302	+0.0130	+0.02646	+0.01483	+0.01265	+0.642	+0.026	+0.2910	+0.0150	+0.0575	−0.0192	−0.00126	+0.377
0.1	+ 0.763	−0.459	+0.906	+0.0274	−0.0291	+0.02469	+0.00533	+0.00860	+1.301	+0.360	+0.2295	+0.0083	+0.0486	+0.0018	+0.00176	+0.620
0.0	+ 1.939	0	+1.000	0	−0.1209	0	0	0	+2.401	+1.000	0	0	−0.0928	0	0	+1.000
$\frac{rt}{l^2} = 0.025$:																
1.6	+ 0.011	+0.019	−0.018	−0.0044	−0.0021	−0.00146	−0.00006	−0.00008	+0.002	+0.009	−0.0118	−0.0031	+0.0065	+0.0088	−0.00706	−0.002
0.8	− 0.150	−0.086	−0.033	−0.0254	+0.0013	−0.01080	+0.01948	+0.00619	−0.080	−0.076	+0.0252	−0.0061	−0.0223	+0.0059	−0.05036	−0.003
0.4	− 0.269	−0.441	+0.336	+0.0031	+0.0237	+0.00736	+0.02650	+0.01307	+0.045	−0.170	+0.2284	+0.0109	−0.0114	−0.0342	−0.02778	+0.111
0.2	+ 0.096	−0.547	+0.722	+0.0241	+0.0122	+0.02384	+0.01433	+0.01273	+0.592	+0.028	+0.3066	+0.0124	+0.0457	−0.0220	−0.00395	+0.356
0.1	+ 0.649	−0.419	+0.909	+0.0221	−0.0243	+0.02262	+0.00520	+0.00892	+1.223	+0.357	+0.2418	+0.0067	+0.0470	−0.0028	+0.00071	+0.603
0.0	+ 1.690	0	+1.000	0	−0.1065	0	0	0	+2.301	+1.000	0	0	−0.0614	0	0	+1.000
$\frac{rt}{l^2} = 0.030$:																
1.6	+ 0.007	+0.014	−0.016	−0.0037	−0.0017	−0.00139	+0.00035	−0.00002	+0.000	+0.006	−0.0099	−0.0025	+0.0053	+0.0076	−0.00683	−0.001
0.8	− 0.127	−0.086	−0.006	−0.0197	+0.0009	−0.00949	+0.01820	+0.00548	−0.071	−0.073	+0.0358	−0.0049	−0.0179	+0.0071	−0.04799	−0.004
0.4	− 0.236	−0.406	+0.354	+0.0025	+0.0203	+0.00624	+0.02497	+0.01251	+0.039	−0.163	+0.2392	+0.0092	−0.0140	−0.0307	−0.02938	+0.099
0.2	+ 0.076	−0.508	+0.728	+0.0199	+0.0115	+0.02149	+0.01370	+0.01280	+0.553	+0.027	+0.3183	+0.0106	+0.0372	−0.0229	−0.00563	+0.338
0.1	+ 0.571	−0.392	+0.911	+0.0184	−0.0204	+0.02074	+0.00502	+0.00920	+1.165	+0.353	+0.2516	+0.0057	+0.0443	−0.0054	+0.00002	+0.588
0.0	+ 1.527	0	+1.000	0	−0.0948	0	0	0	+2.233	+1.000	0	0	−0.0424	0	0	+1.000

TABLE 3A.—(Continued)

$\frac{s}{\sqrt[4]{r t l^2}}$	Tangential Edge Load				Radial Edge Load				Shear Edge Load				Moment Edge Load			
	(c) (Continued)															
	T_x	S	$T\phi$	$M\phi$	T_x	S	$T\phi$	$M\phi$	T_x	S	$T\phi$	$M\phi$	T_x	S	$T\phi$	$M\phi$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
$\frac{r t}{l^2} = 0.040$:																
1.6	+ 0.002	+0.008	−0.011	−0.0025	−0.0011	−0.00114	+0.00069	+0.00003	−0.001	+0.003	−0.0065	−0.0017	+0.0037	+0.0056	−0.00595	−0.001
0.8	− 0.099	−0.084	+0.027	−0.0134	+0.0001	−0.00756	+0.01547	+0.00448	−0.058	−0.068	+0.0483	−0.0036	−0.0119	+0.0080	−0.04220	−0.006
0.4	− 0.198	−0.361	+0.373	+0.0013	+0.0155	+0.00439	+0.02195	+0.01167	+0.026	−0.154	+0.2511	+0.0069	−0.0156	−0.0248	−0.02064	+0.081
0.2	+ 0.053	−0.461	+0.732	+0.0142	+0.0104	+0.01764	+0.01241	+0.01292	+0.494	+0.021	+0.3343	+0.0082	+0.0258	−0.0224	−0.00724	+0.310
0.1	+ 0.483	−0.362	+0.911	+0.0135	−0.0148	+0.01761	+0.00462	+0.00965	+1.083	+0.340	+0.2668	+0.0043	+0.0381	−0.0078	−0.00072	+0.565
0.0	+ 1.351	0	+1.000	0	−0.0771	0	0	0	+2.150	+1.000	0	0	−0.0223	0	0	+1.000
$\frac{r t}{l^2} = 0.060$:																
1.6	− 0.000	+0.002	−0.005	−0.0013	−0.0006	−0.00069	+0.00066	+0.00005	−0.001	+0.000	−0.0023	−0.0009	+0.0021	+0.0031	−0.00397	−0.001
0.8	− 0.071	−0.072	+0.052	−0.0079	−0.0005	−0.00521	+0.01099	+0.00330	−0.046	−0.058	+0.0549	−0.0025	−0.0057	+0.0077	−0.03157	−0.006
0.4	− 0.168	−0.309	+0.375	+0.0001	+0.0100	+0.00207	+0.01705	+0.01054	+0.005	−0.146	+0.2546	+0.0043	−0.0142	−0.0165	−0.02659	+0.058
0.2	+ 0.012	−0.415	+0.725	+0.0091	+0.0086	+0.01245	+0.01024	+0.01303	+0.415	+0.003	+0.3503	+0.0057	+0.0141	−0.0189	−0.00797	+0.271
0.1	+ 0.375	−0.337	+0.907	+0.0092	−0.0085	+0.01326	+0.00394	+0.01027	+0.983	+0.314	+0.2867	+0.0030	+0.0282	−0.0084	−0.00133	+0.530
0.0	+ 1.165	0	+1.000	0	−0.0554	0	0	0	+2.076	+1.000	0	0	−0.0081	0	0	+1.000
$\frac{r t}{l^2} = 0.080$:																
1.6	− 0.001	+0.001	−0.002	−0.0008	−0.0003	−0.00043	+0.00047	+0.00004	−0.001	−0.000	−0.0007	−0.0005	+0.0013	+0.0020	−0.00259	−0.000
0.8	− 0.056	−0.062	+0.056	−0.0055	−0.0007	−0.00387	+0.00803	+0.00263	−0.038	−0.050	+0.0527	−0.0020	−0.0029	+0.0068	−0.02401	−0.006
0.4	− 0.155	−0.280	+0.362	−0.0005	+0.0070	+0.00084	+0.01368	+0.00975	−0.010	−0.141	+0.2485	+0.0029	−0.0121	−0.0115	−0.02306	+0.044
0.2	− 0.012	−0.396	+0.712	+0.0065	+0.0073	+0.00933	+0.00865	+0.01305	+0.362	−0.015	+0.3569	+0.0044	+0.0084	−0.0158	−0.00764	+0.244
0.1	+ 0.323	−0.331	+0.901	+0.0069	−0.0052	+0.01054	+0.00342	+0.01069	+0.919	+0.289	+0.2997	+0.0024	+0.0218	−0.0078	−0.00145	+0.505
0.0	+ 1.097	0	+1.000	0	−0.0428	0	0	0	+2.045	+1.000	0	0	−0.0037	0	0	+1.000
$\frac{r t}{l^2} = 0.100$:																
1.6	− 0.001	−0.000	−0.000	−0.0005	−0.0002	−0.00028	+0.00033	+0.00003	−0.001	−0.000	−0.0001	−0.0003	+0.0008	+0.0013	−0.00173	−0.000
0.8	− 0.047	−0.054	+0.053	−0.0042	−0.0007	−0.00300	+0.00609	+0.00219	−0.033	−0.043	+0.0484	−0.0016	−0.0015	+0.0059	−0.01878	−0.006
0.4	− 0.146	−0.260	+0.346	−0.0009	+0.0052	+0.00017	+0.01131	+0.00913	−0.020	−0.138	+0.2398	+0.0021	−0.0104	−0.0084	−0.02088	+0.035
0.2	− 0.029	−0.385	+0.698	+0.0050	+0.0063	+0.00730	+0.00750	+0.01302	+0.323	−0.030	+0.3594	+0.0036	+0.0051	−0.0135	−0.00706	+0.224
0.1	+ 0.289	−0.330	+0.895	+0.0056	−0.0033	+0.00870	+0.00303	+0.01100	+0.872	+0.268	+0.3091	+0.0020	+0.0174	−0.0072	−0.00135	+0.485
0.0	+ 1.064	0	+1.000	0	−0.0348	0	0	0	+2.029	+1.000	0	0	−0.0020	0	0	+1.000

TABLE 3B.—Edge Displacements of Simply Supported Cylindrical Shells

(a) Basic Formulas

	Tangential Edge Load	Radial Edge Load	Shear Edge Load	Moment Edge Load
Tangential Displacement v (Inward Positive)—	$T_L \times \frac{l}{E t} \times$ Col. (1) $\times \sin \frac{n \pi x}{l}$	$R_L \times \frac{l^2}{E t^2} \times$ Col. (4) $\times \sin \frac{n \pi x}{l}$	$S_L \times \frac{l}{E t} \times$ Col. (7) $\times \sin \frac{n \pi x}{l}$	$M_L \times \frac{l}{E t^2} \times$ Col. (10) $\times \sin \frac{n \pi x}{l}$
Radial Displacement w (Outward Positive)—	$T_L \times \frac{l^2}{E t^2} \times$ Col. (2) $\times \sin \frac{n \pi x}{l}$	$R_L \times \frac{l^3}{E t^3} \times$ Col. (5) $\times \sin \frac{n \pi x}{l}$	$S_L \times \frac{l^2}{E t^2} \times$ Col. (8) $\times \sin \frac{n \pi x}{l}$	$M_L \times \frac{l^2}{E t^3} \times$ Col. (11) $\times \sin \frac{n \pi x}{l}$
Rotation θ (Counter Clockwise Positive)—	$T_L \times \frac{l t}{E I} \times$ Col. (3) $\times \sin \frac{n \pi x}{l}$	$R_L \times \frac{l^2}{E I} \times$ Col. (6) $\times \sin \frac{n \pi x}{l}$	$S_L \times \frac{l t}{E I} \times$ Col. (9) $\times \sin \frac{n \pi x}{l}$	$M_L \times \frac{l}{E I} \times$ Col. (12) $\times \sin \frac{n \pi x}{l}$

(b) $n = 1$

$\frac{r t}{l^2}$	v (1)	w (2)	θ (3)	v (4)	w (5)	θ (6)	v (7)	w (8)	θ (9)	v (10)	w (11)	θ (12)
0.002	−77.30	−1.495	0.5712	1.495	0.03568	−0.01782	−11.91	−0.1861	0.05752	6.854	0.2138	−0.1710
0.003	−54.76	−1.409	0.4742	1.409	0.04535	−0.02027	− 9.393	−0.1943	0.05227	5.691	0.2433	−0.1795
0.004	−42.58	−1.342	0.4118	1.342	0.05332	−0.02206	− 7.909	−0.1990	0.04830	4.942	0.2648	−0.1851
0.006	−29.98	−1.243	0.3329	1.243	0.06635	−0.02459	− 6.176	−0.2041	0.04246	3.995	0.2951	−0.1920
0.008	−23.24	−1.169	0.2830	1.169	0.07690	−0.02634	− 5.164	−0.2062	0.03819	3.396	0.3161	−0.1962
0.010	−19.03	−1.110	0.2477	1.110	0.08582	−0.02764	− 4.485	−0.2069	0.03484	2.973	0.3317	−0.1990
0.015	−13.18	−1.000	0.1911	1.000	0.1036	−0.02983	− 3.452	−0.2058	0.02878	2.293	0.3580	−0.2032
0.020	−10.13	−0.9204	0.1565	0.9204	0.1174	−0.03120	− 2.853	−0.2029	0.02457	1.878	0.3744	−0.2055
0.025	− 8.241	−0.8582	0.1327	0.8582	0.1286	−0.03213	− 2.455	−0.1994	0.02142	1.593	0.3855	−0.2069
0.030	− 6.958	−0.8074	0.1152	0.8074	0.1381	−0.03279	− 2.168	−0.1956	0.01894	1.383	0.3935	−0.2078
0.040	− 5.320	−0.7277	0.09096	0.7277	0.1533	−0.03365	− 1.776	−0.1879	0.01525	1.091	0.4038	−0.2090
0.060	− 3.643	−0.6171	0.06322	0.6171	0.1750	−0.03447	− 1.333	−0.1735	0.01062	0.7586	0.4136	−0.2100
0.080	− 2.791	−0.5407	0.04772	0.5407	0.1900	−0.03477	− 1.084	−0.1608	0.007822	0.5726	0.4172	−0.2105
0.100	− 2.278	−0.4832	0.03784	0.4832	0.2011	−0.03487	− 0.9241	−0.1495	0.005967	0.4498	0.4184	−0.2108

TABLE 3B.—(Continued)

$\frac{rt}{l^2}$	Tangential Edge Load			Radial Edge Load			Shear Edge Load			Moment Edge Load		
	v (1)	w (2)	θ (3)	v (4)	w (5)	θ (6)	v (7)	w (8)	θ (9)	v (10)	w (11)	θ (12)
(c) $n = 3$												
0.002	− 3.719	−0.1055	0.05620	0.1055	0.004158	−0.003413	− 1.020	−0.02269	0.008700	0.6745	0.04096	−0.06824
0.003	− 2.558	−0.09298	0.04171	0.09298	0.004911	−0.003602	− 0.7767	−0.02198	0.006788	0.5005	0.04322	−0.06911
0.004	− 1.957	−0.08409	0.03313	0.08409	0.005470	−0.003707	− 0.6371	−0.02122	0.005522	0.3975	0.04448	−0.06954
0.006	− 1.340	−0.07172	0.02324	0.07172	0.006274	−0.003811	− 0.4789	−0.01974	0.003916	0.2789	0.04573	−0.06994
0.008	− 1.025	−0.06316	0.01767	0.06316	0.006836	−0.003854	− 0.3899	−0.01841	0.002931	0.2120	0.04624	−0.07013
0.010	− 0.8354	−0.05668	0.01409	0.05668	0.007256	−0.003871	− 0.3322	−0.01722	0.002269	0.1691	0.04645	−0.07023
0.015	− 0.5836	−0.04543	0.009073	0.04543	0.007958	−0.003871	− 0.2494	−0.01475	0.001309	0.1089	0.04646	−0.07038
0.020	− 0.4615	−0.03799	0.006510	0.03799	0.008385	−0.003855	− 0.2057	−0.01283	0.000820	0.07812	0.04627	−0.07046
0.025	− 0.3910	−0.03260	0.004998	0.03260	0.008666	−0.003839	− 0.1793	−0.01130	0.000543	0.05996	0.04607	−0.07052
0.030	− 0.3466	−0.02851	0.004019	0.02851	0.008861	−0.003824	− 0.1620	−0.01005	0.000375	0.04823	0.04589	−0.07056
0.040	− 0.2957	−0.02270	0.002709	0.02270	0.009104	−0.003803	− 0.1433	−0.008179	0.000197	0.03251	0.04563	−0.07061
0.060	− 0.2528	−0.01598	0.001791	0.01598	0.009326	−0.003780	− 0.1236	−0.005873	0.000071	0.02149	0.04536	−0.07067
0.080	− 0.2360	−0.01226	0.001305	0.01226	0.009420	−0.003769	− 0.1164	−0.004542	0.000033	0.01566	0.04523	−0.07070
0.100	− 0.2277	−0.00992	0.001027	0.00992	0.009467	−0.003764	− 0.1129	−0.003690	0.000017	0.01232	0.04516	−0.07071

E.2 ADDITIONAL TABLES FOR DESIGN OF CYLINDRICAL SHELLS

1. Coefficients for Design of Cylindrical Concrete Shells—Portland Cement Association U.S.A.—1959.
2. Design constants for Interior Cylindrical Concrete Shells—Portland Cement Association U.S.A.—(Advanced Engineering Bulletin No. 1)—1960.

E.3 OTHER PUBLISHED TABLES FOR DESIGN OF REINFORCED CONCRETE SHELLS

1. Design of Parabolic Conoidal Shells. "Reinforced Concrete Conoidal Shell Roofs". Flexural Theory Design Tables by C.B. Wilby and M.M. Nadvi, published by Cement and Concrete Association, U.K.
2. Design of Elliptical Paraboloids. Coefficients for computing stress resultants in Elliptic paraboloids. Refer "Shells of Double Curvature by Parme A.L. Trans ASCE, Vol. 123 (1958), page 980. They are also available in the book, *Thin Concrete Shell Structures* by D.P. Billington, page 258.
3. Hyperbolic paraboloid Groined vaults. Refer Elementary Analysis of Hyperbolic Paraboloids Shells by A.L. Parme Bulletin 4. The International Association of shell structures Madrid, 1960. Reprint No. 51. Indian Concrete Journal, 1963.

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P.C. Varghese

This comprehensive and well-organized text provides a masterly exposition of the fundamentals of analysis and design of reinforced concrete shells and folded plates, commonly known as **thin concrete roof structures**. Divided into 20 chapters, the book presents practical designs of different types of domes, cylindrical shells, paraboloids, conoids, and groined shells, as well as various types of folded plates. The text also incorporates tables from ASCE Manual No. 31.

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